CHAPTER 1 First-Order Differential Equations

1.1 Dynamical Systems: Modeling

■ Constants of Proportionality

1. $\dfrac{dA}{dt} = kA \qquad (k < 0)$

2. $\dfrac{dA}{dt} = kA \qquad (k < 0)$

3. $\dfrac{dP}{dt} = kP(20,000 - P)$

4. $\dfrac{dA}{dt} = \dfrac{kA}{\sqrt{t}}$

5. $\dfrac{dG}{dt} = \dfrac{kN}{A}$

■ A Walking Model

6. Because $d = \upsilon t$ where d = distance traveled, υ = average velocity, and t = time elapsed, we have the model for the time elapsed as simply the equation $t = \dfrac{d}{\upsilon}$. Now, if we measure the distance traveled as 1 mile and the average velocity as 3 miles/hour, then our model predicts the time to be $t = \dfrac{d}{\upsilon} = \dfrac{1}{3}\,\text{hr}$, or 20 minutes. If it actually takes 20 minutes to walk to the store, the model is perfectly accurate. This model is so simple we generally don't even think of it as a model.

■ A Falling Model

7. (a) Galileo has given us the model for the distance $s(t)$ a ball falls in a vacuum as a function of time t: On the surface of the earth the acceleration of the ball is a constant, so $\dfrac{d^2s}{dt^2} = g$, where $g \approx 32.2\ \text{ft}/\text{sec}^2$. Integrating twice and using the conditions $s(0) = 0$, $\dfrac{ds(0)}{dt} = 0$, we find

$$s(t) = \frac{1}{2}gt^2\, s(t) = \frac{1}{2}gt^2 .$$

(b) We find the time it takes for the ball to fall 100 feet by solving for t the equation $100 = \frac{1}{2}gt^2 = 16.1t^2$, which gives $t = 2.49$ seconds. (We use 3 significant digits in the answer because g is also given to 3 significant digits.)

(c) If the observed time it takes for a ball to fall 100 feet is 2.6 seconds, but the model predicts 2.49 seconds, the first thing that might come to mind is the fact that Galileo's model assumes the ball is falling in a vacuum, so some of the difference might be due to air friction.

■ The Malthus Rate Constant *k*

8. (a) Replacing

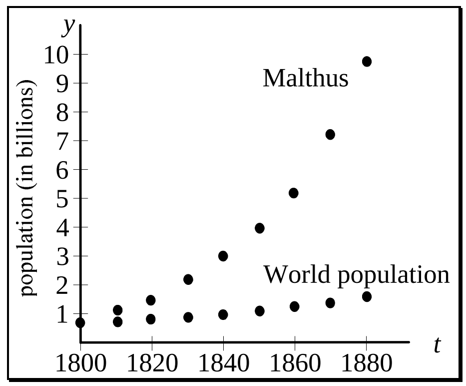
$$e^{0.03} \approx 1.03045$$

in Equation (3) gives

$$y = 0.9(1.03045)^t,$$

which increases roughly 3% per year.

(b)



(c) Clearly, Malthus' rate estimate was far too high. The world population indeed rises, as does the exponential function, but at a far slower rate.

If $y(t) = 0.9e^{rt}$, you might try solving $y(200) = 0.9e^{200r} = 6.0$ for r. Then

$$200r = \ln\frac{6}{0.9} \approx 1.897$$

so

$$r \approx \frac{1.897}{200} \approx 0.0095,$$

which is less than 1%.

■ Population Update

9. (a) If we assume the world's population in billions is currently following the unrestricted growth curve at a rate of 1.7% and start with the UN figure for 2000, then

$$y_0 e^{kt} = 6.056e^{0.017t},$$

and the population in the years 2010 ($t = 10$), 2020 ($t = 20$), and 2030 ($t = 30$), would be, respectively, the values

$$6.056e^{0.017(10)} = 7.176$$
$$6.056e^{0.017(20)} \approx 8.509$$
$$6.056e^{0.017(30)} \approx 10.083.$$

These values increasingly exceed the United Nations predictions so the U.N. is assuming a growth rate less than 1.7%.

(b) 2010: $6.056e^{10r} = 6.843$

$$e^{10r} = \frac{6.843}{6.056} = 1.13$$
$$10r = \ln(1.13) = 0.1222$$
$$r = 1.2\%$$

2020: $6843e^{10r} = 7568$

$$e^{10r} = \frac{7.578}{6.843} = 1.107$$
$$10r = \ln(1.107) = 0.102$$
$$r = 1.0\%$$

2030: $7.578e^{10r} = 8.199$

$$e^{10r} = \frac{8.199}{7.578} = 1.082$$
$$10r = \ln(1.082) = 0.079$$
$$r = 0.8\%$$

■ The Malthus Model

10. (a) Malthus thought the human population was increasing exponentially e^{kt}, whereas the food supply increases arithmetically according to a linear function $a + bt$. This means the number of people per food supply would be in the ratio $\frac{e^{kt}}{(a+bt)}$, which although not a pure exponential function, is concave up. This means that the rate of increase in the number of persons per the amount of food is increasing.

(b) The model cannot last forever since its population approaches infinity; reality would produce some limitation. The exponential model does not take under consideration starvation, wars, diseases, and other influences that slow growth.

(c) A linear growth model for food supply will increase supply without bound and fails to account for technological innovations, such as mechanization, pesticides and genetic engineering. A nonlinear model that approaches some finite upper limit would be more appropriate.

(d) An exponential model is sometimes reasonable with simple populations over short periods of time, e.g., when you get sick a bacteria might multiply exponentially until your body's defenses come into action or you receive appropriate medication.

■ Discrete-Time Malthus

11. (a) Taking the 1798 population as $y_0 = 0.9$ (0.9 billion), we have the population in the years 1799, 1800, 1801, and 1802, respectively

$$y_1 = 1.03(0.9) = 0.927$$
$$y_2 = (1.03)^2(0.9) = 0.956$$
$$y_3 = (1.03)^3(0.9) = 0.983$$
$$y_4 = (1.03)^4(0.9) = 1.023.$$

(b) In 1990 we have $t = 192$, hence

$$y_{192} = (1.03)^{192}(0.9) \approx 262 \text{ (262 billion).}$$

(c) The discrete model will always give a value lower than the continuous model. Later, when we study compound interest, you will learn the exact relationship between discrete compounding (as in the discrete-time Malthus model) and continuous compounding (as described by $y' = ky$).

■ Verhulst Model

12. $\dfrac{dy}{dt} = y(k - cy)$. The constant k affects the initial growth of the population whereas the constant c controls the damping of the population for larger y. There is no reason to suspect the two values would be the same and so a model like this would seem to be promising if we only knew their values. From the equation $y' = y(k - cy)$, we see that for small y the population closely obeys $y' = ky$, but reaches a steady state $(y' = 0)$ when $y = \dfrac{k}{c}$.

■ Suggested Journal Entry

13. Student Project

1.2 Solutions and Direction Fields

[redacted]

■ Verification

1. If $y = 2\tan 2t$, then $y' = 4\sec^2 2t$. Substituting y' and y into $y' = y^2 + 4$ yields a trigonometric identity

$$4\sec^2(2t) \equiv 4\tan^2(2t) + 4.$$

2. Substituting

$$y = 3t + t^2$$
$$y' = 3 + 2t$$

into $y' = \frac{1}{t}y + t$ yields the identity

$$3 + 2t \equiv \frac{1}{t}(3t + t^2) + t.$$

3. Substituting

$$y = t^2 \ln t$$
$$y' = 2t\ln t + t$$

into $y' = \frac{2}{t}y + t$ yields the identity

$$2t\ln t + t \equiv \frac{2}{t}\left(t^2 \ln t\right) + t.$$

4. If $y = \int_0^t e^{-2\left(s^2 - t^2\right)} ds = e^{2t^2}\int_0^t e^{-2s^2} ds$, then, using the product rule and the fundamental theorem of calculus, we have

$$y' = e^{2t^2}e^{-2t^2} + 4te^{2t^2}\int_0^t e^{-2s^2} ds = 1 + 4te^{2t^2}\int_0^t e^{-2s^2} ds.$$

Substituting y' and y into $y' - 4ty$ yields

$$1 + 4te^{2t^2}\int_0^t e^{-2s^2} ds - 4te^{2t^2}\int_0^t e^{-2s^2} ds,$$

which is 1 as the differential equation requires.

■ **IVPs**

5. Here

$$y = \frac{1}{2}e^{-t} - e^{-3t}$$
$$y' = -\frac{1}{2}e^{-t} + 3e^{-3t}.$$

Substituting into the differential equation

$$y' + 3y = e^{-t}$$

we get

$$\left(-\frac{1}{2}e^{-t} + 3e^{-3t}\right) + 3\left(\frac{1}{2}e^{-t} - e^{-3t}\right),$$

which is equal to e^{-t} as the differential equation requires. It is also a simple matter to see that $y(0) = -\frac{1}{2}$, and so the initial condition is also satisfied.

6. Another direct substitution

■ **Applying Initial Conditions**

7. If $y = ce^{t^2}$, then we have $y' = 2cte^{t^2}$ and if we substitute y and y' into $y' = 2ty$, we get the identity $2cte^{t^2} \equiv 2t\left(ce^{t^2}\right)$. If $y(0) = 2$, then we have $ce^{0^2} \equiv c = 2$.

8. We have

$$y = e^t \cos t + ce^t$$
$$y' = e^t \cos t - e^t \sin t + ce^t$$

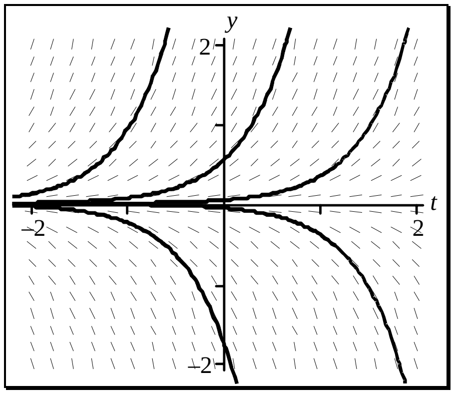
and substituting y and y' into $y' - y$ yields

$$\left(e^t \cos t - e^t \sin t + ce^t\right) - \left(e^t \cos t + ce^t\right),$$

which is $-e^t \sin t$. If $y(0) = -1$, then $-1 = e^0 \cos 0 + ce^0$ yields $c = -2$.

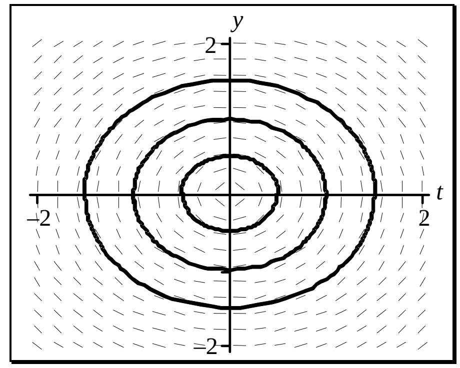
■ Using the Direction Field

9. $y' = 2y$



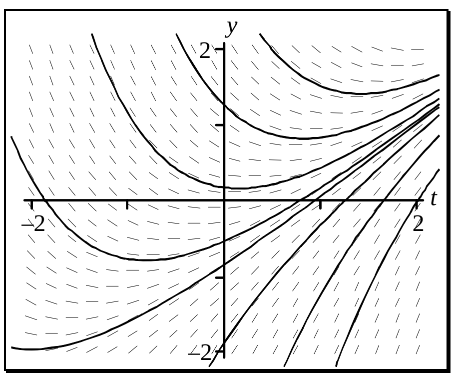
Solutions are $y = ce^{2t}$.

10. $y' = -\dfrac{t}{y}$



Solutions are $y = \sqrt{c - t^2}$.

11. $y' = t - y$



Solutions are $y = t - 1 + ce^{-t}$.

■ Linear Solution

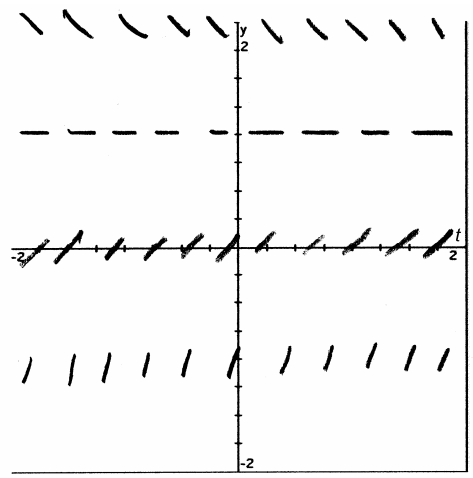
12. It appears from the direction field that there is a straight-line solution passing through (0, –1) with slope 1, i.e., the line $y = t - 1$. Computing $y' = 1$, we see it satisfies the DE $y' = t - y$ because $1 \equiv t - (t - 1)$.

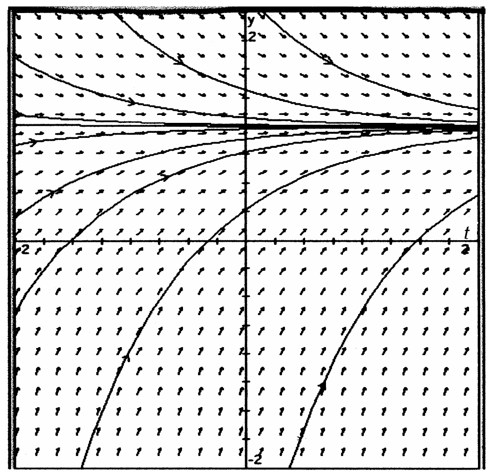
Stability

13. $y' = 1 - y = 0$

When $y = 1$, the direction field shows a *stable* equilibrium solution.

For $y > 1$, slopes are negative; for $y < 1$, slopes are positive.

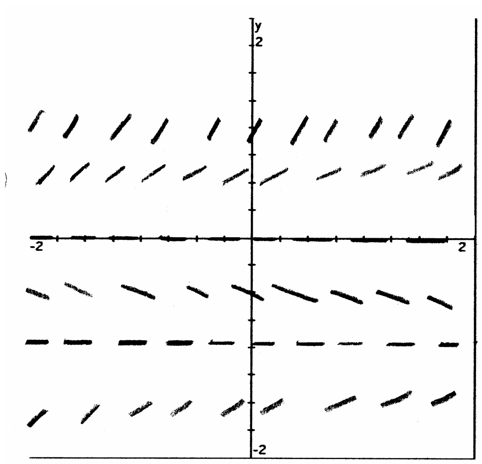


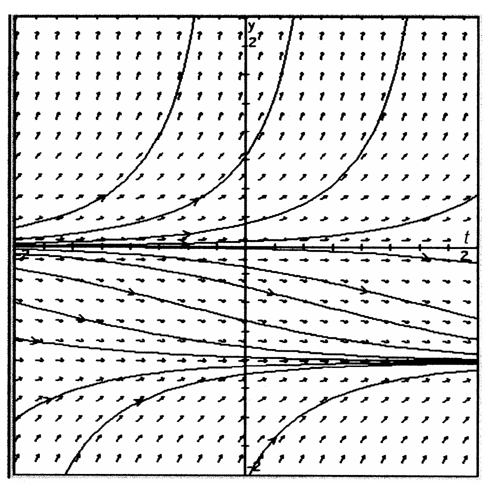


14. $y' = y(y+1) = 0$

When $y = 0$, an *unstable* equilibrium solution exists, and when $y = -1$, a *stable* equilibrium solution exists.

For
$$y = 3, \quad y' = 3(4) = 12$$
$$y = 1, \quad y' = 1(2) = 2$$
$$y = -\frac{1}{2}, \quad y' = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$
$$y = -2, \quad y' = (-2)(-1) = 2$$
$$y = 4, \quad y' = (-4)(-3) = 12$$





15. $y' = t^2(1 - y^2)$

Two equilibrium solutions:

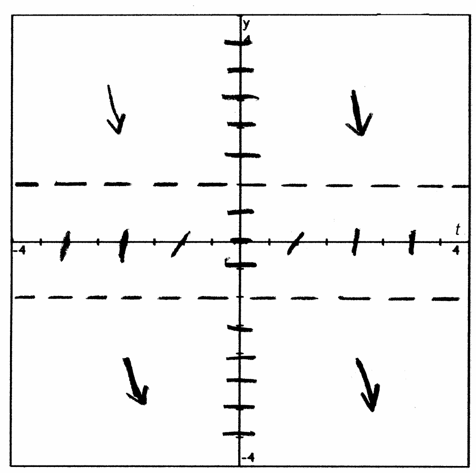
$y = 1$ is stable

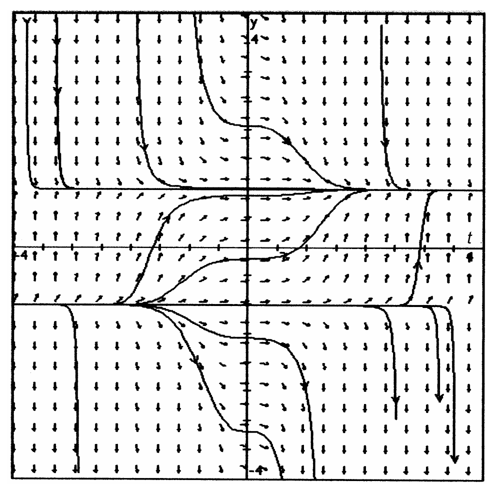
$y = -1$ is unstable

Between the equilibria the slopes (positive) are shallower as they are located further from the horizontal axis.

Outside the equilibria the slopes are negative and become steeper as they are found further from the horizontal axis.

All slopes become steeper as they are found further from the vertical axis.





■ **Match Game**

16. (C) Because the slope is always the same

17. (D) Because the slope is always the value of y

18. (F) Because F is the only direction field that has vertical slopes when $t = 0$ and zero slopes when $y = 0$

19. (B) Because it is the only direction field that has all zero slopes when $t = 0$

20. (E) The slope is always positive and equal to the square of the distance from the origin.

21. (A) Because it is undefined when $t = 0$ and the directional field has slopes that are independent of y, with the same sign as that of t

■ Concavity

22. $y' = y^2 - 4$

$y'' = 2yy' = 2y(y+2)(y-2)$

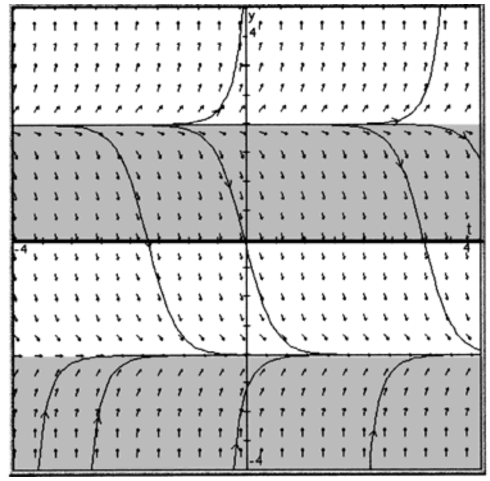
When $y = 0$, we find inflection points for solutions.

Equilibrium solutions occur when $y = 2$ (*unstable*) or when $y = -2$ (*stable*).

Solutions are

concave *up* for $y > 2$, and $y \in (-2, 0)$;

concave *down* for $y < -2$, and $y \in (0, 2)$



Horizontal axis is locus of inflection points; shaded regions are where solutions are concave *down*.

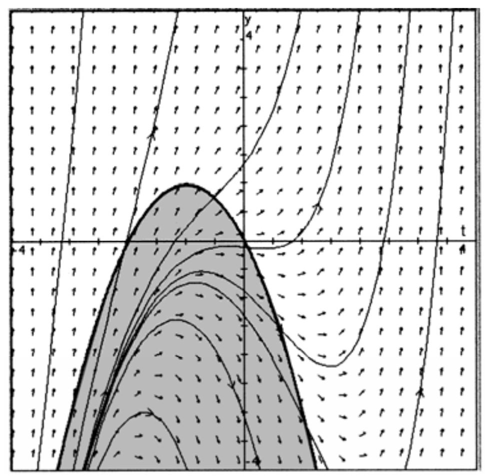
23. $y' = y + t^2$

$y'' = y' + 2t = y + t^2 + 2t = 0$

When $y = -t^2 - 2t$, $y'' = 0$, so

we have a locus of inflection points.

Solutions are concave *up* above the parabola of inflection points, concave *down* below.



Parabola is locus of inflection points; shaded regions are where solutions are concave *down*.

24. $y' = y^2 - t$

$y'' = 2yy' - 1$

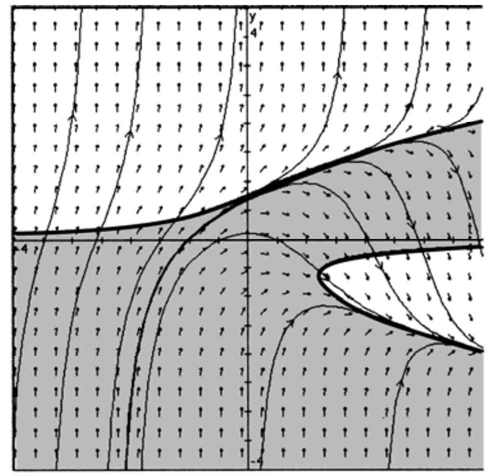
$= 2y^3 - 2yt - 1 = 0$

When $\underbrace{t = \frac{2y^3 - 1}{2y} = y^2 - \frac{1}{2y}}$, then $y'' = 0$

and we have a locus of inflection points.

The locus of inflection points has two branches: Above the upper branch, and to the right of the lower branch, solutions are concave *up*.

Below the upper branch but outside the lower branch, solutions are concave *down*.



Bold curves are the locus of inflection points; shaded regions are where solutions are concave *down*.

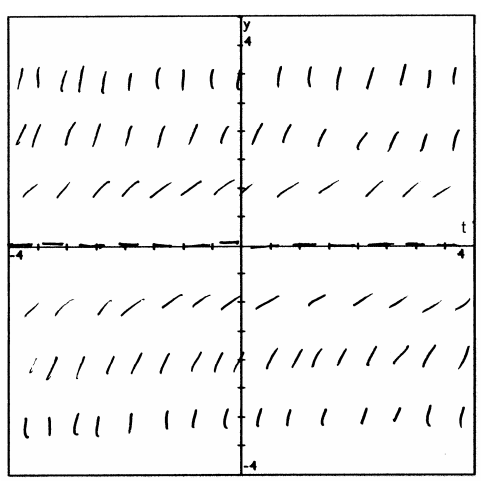
■ Asymptotes

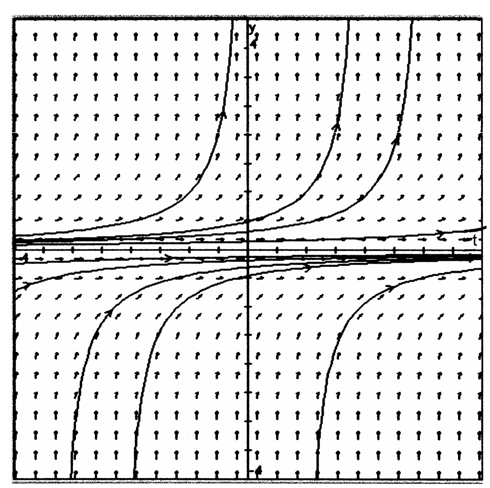
25. $y' = y^2$

Because y' depends only on y, isoclines will be horizontal lines, and solutions will be horizontal translates.

Slopes get steeper ever more quickly as distance from the x-axis increases.

If the y-axis extends high enough, you may suspect (correctly) that undefined solutions will each have a (different) vertical asymptote. When slopes are increasing quickly, it's a good idea to check *how* fast. The direction field will give good intuition, if you look far enough.
Compare with $y' = y$ for a case where the solutions do not have asymptotes.

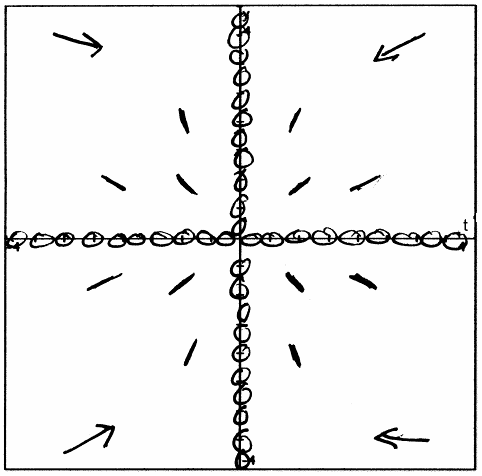


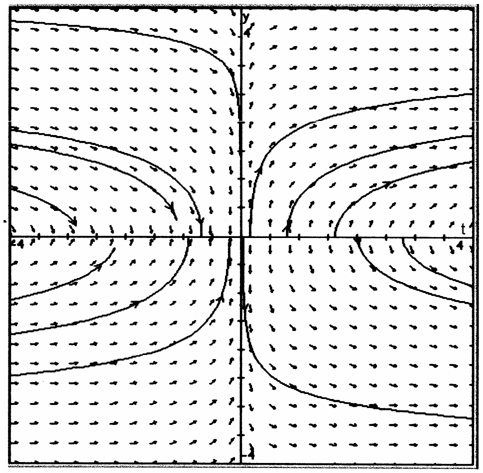


26. $y' = \dfrac{1}{ty}$

The DE is undefined for $t = 0$ or $y = 0$, *so solutions do not cross either axis.*

However, as solutions approach or depart from the horizontal axis, they asymptotically approach a vertical slope.



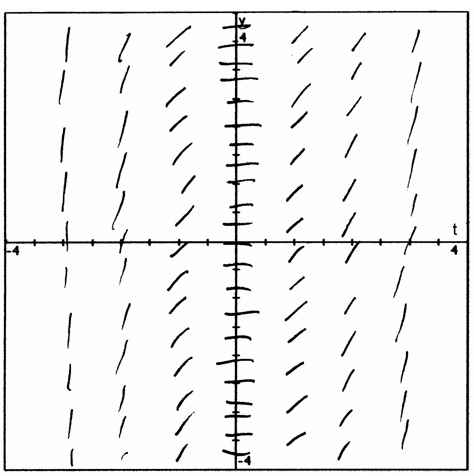


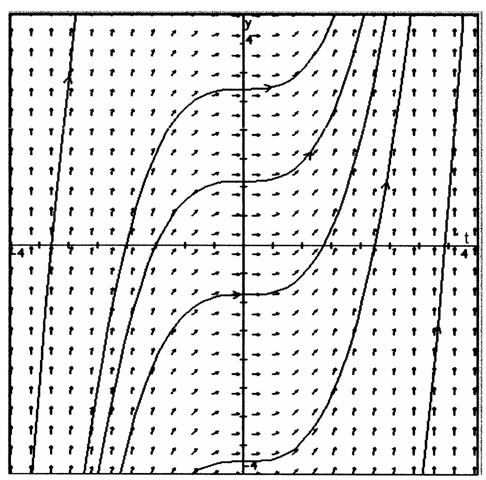
Every solution has a vertical asymptote
when it is close to the horizontal axis.

27. $y' = t^2$

There are *no* asymptotes.

As $t \to \infty$ (or $t \to -\infty$) slopes get steeper and steeper, but they do not actually approach vertical for any finite value of t.

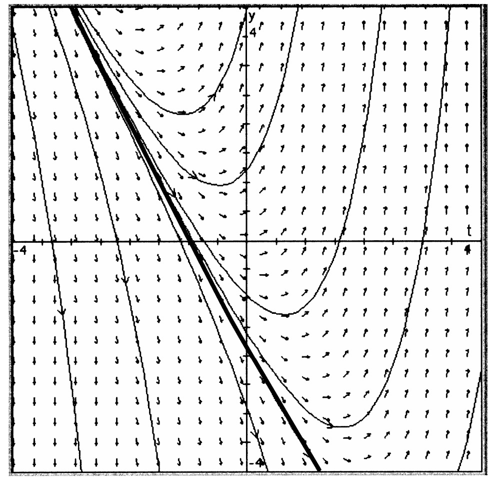




No asymptote

28. $y' = 2t + y$

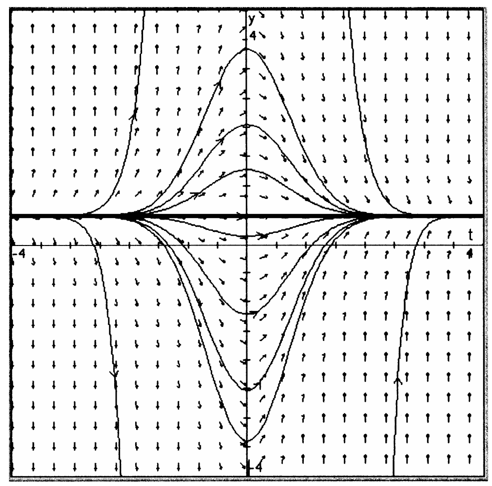
Solutions to this DE have an oblique asymptote–they all curve away from it as $t \to \infty$, moving down then up on the right, simply down on the left. The equation of this asymptote can be at least approximately read off the graphs as $y = -2t - 2$. In fact, you can verify that this line satisfies the DE, so this asymptote is also a solution.



Oblique Asymptote

29. $y' = -2ty + t$

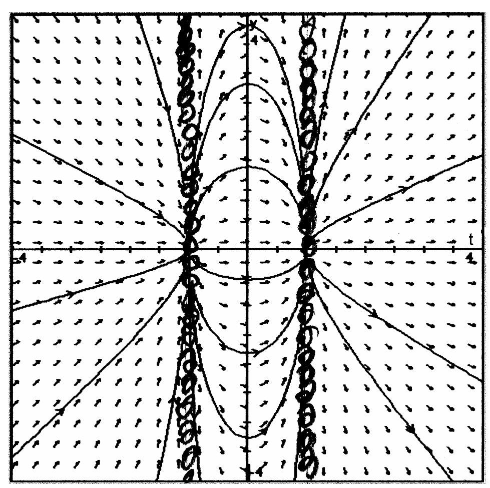
Here we have a horizontal asymptote,
at $t = \frac{1}{2}$.



Horizontal asymptote

30. $y' = \dfrac{ty}{t^2 - 1}$

At $t = 1$ and $t = -1$ the DE is undefined. The direction field shows that as $y \to 0$ from either above or below, solutions asymptotically approach vertical slope. However, $y = 0$ is a solution to the DE, and the *other solutions do not cross* the horizontal axis for $t \neq \pm 1$. (See Picard's Theorem Sec. 1.5.)

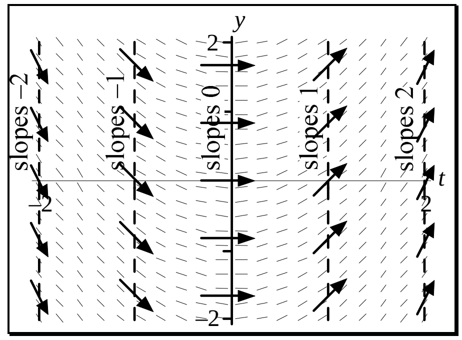


Vertical asymptotes for $t \to 1$ or $t \to -1$

■ Isoclines

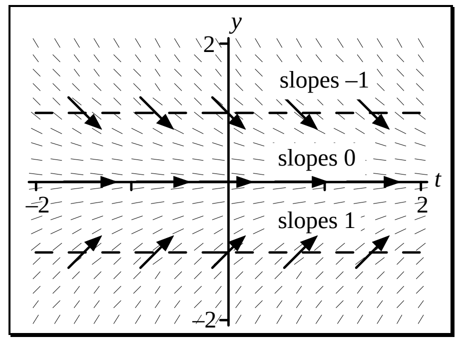
31. $y' = t$.

The isoclines are vertical lines $t = c$, as follows for $c = 0,\ \pm 1,\ \pm 2$ shown in the figure.



32. $y' = -y$.

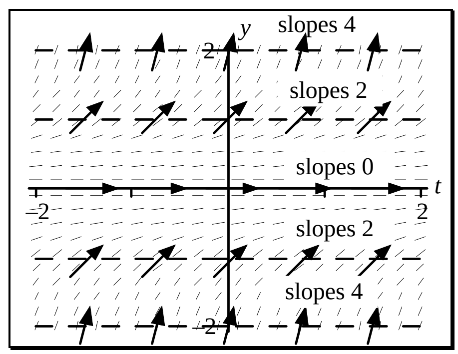
Here the slope of the solution is negative when $y > 0$ and positive for $y < 0$. The isoclines for $c = -1,\ 0,\ 1$ are shown in the figure.



33. $y' = y^2$.

Here the slope of the solution is always ≥ 0.

The isoclines where the slope is $c > 0$ are the horizontal lines $y = \pm\sqrt{c} \geq 0$. In other words the isoclines where the slope is 4 are $y = \pm 2$. The isoclines for $c = 0$, 2, and 4 are shown in the figure.

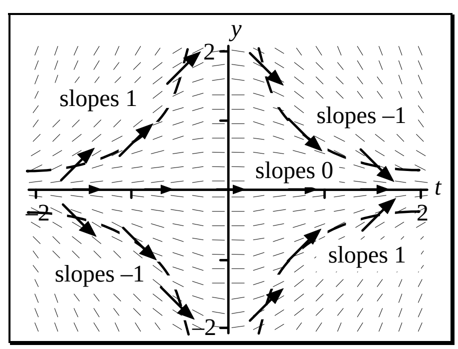


34. $y' = -ty$.

Setting $-ty = c$, we see that the points where the slope is c are along the curve $y = -\dfrac{c}{t}$, $t \neq 0$ or hyperbolas in the ty plane.

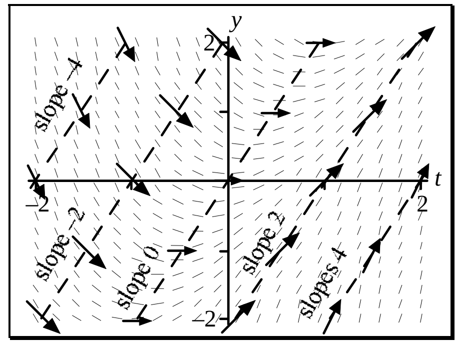
For $c = 1$, the isocline is the hyperbola $y = -\dfrac{1}{t}$.

For $c = -1$, the isocline is the hyperbola $y = \dfrac{1}{t}$.

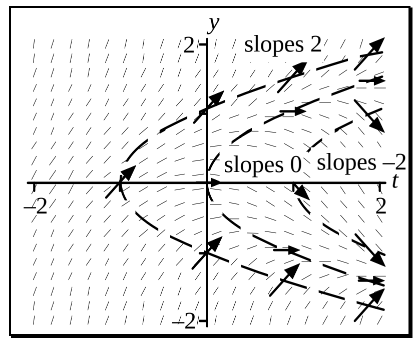


When $t = 0$ the slope is zero for any y; when $y = 0$ the slope is zero for any t, and $y = 0$ is in fact a solution. See figure for the direction field for this equation with isoclines for $c = 0,\ \pm 1$.

35. $y' = 2t - y$. The isocline where $y' = c$ is the straight line $y = 2t - c$. The isoclines with slopes $c = -4$, –2, 0, 2, 4 are shown from left to right (see figure).



36. $y' = y^2 - t$. The isocline where $y' = c$ is a parabola that opens to the right. Three isoclines, with slopes $c = 2$, 0, –2, are shown from left to right (see figure).



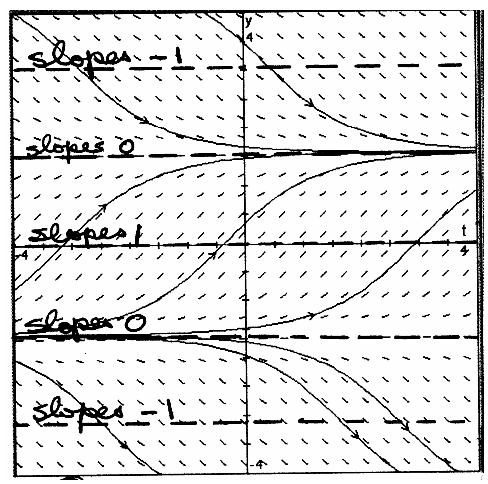
37. $y' = \cos y$

$$y' = c = \begin{cases} 0 \text{ when } y = \text{odd multiples of } \dfrac{\pi}{2} \\ 1 \text{ when } y = 0,\ 2\pi,\ 4\pi,\ \ldots \\ -1 \text{ when } y = \pi,\ 3\pi,\ \ldots \end{cases}$$

Additional observations:

$|y'| \le 1$ for all y.

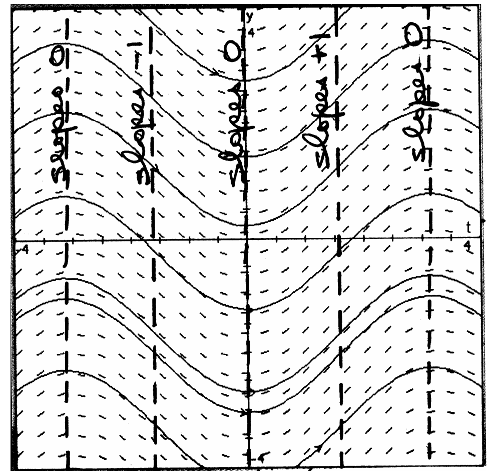
When $y = \dfrac{\pi}{4}$, this information produces a slope field in which the constant solutions, at $y = (2n+1)\dfrac{\pi}{2}$, act as horizontal asymptotes.



38. $y' = \sin t$

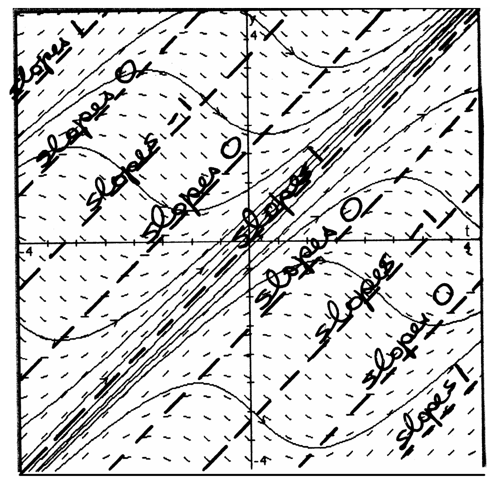
$$y' = c = \begin{cases} 0 \text{ when } t = 0, \pi, 2\pi, \dots \\ 1 \text{ when } t = -\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dots \\ -1 \text{ when } t = -\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dots \end{cases}$$

The direction field indicates oscillatory periodic solutions, which you can verify as $y = -\cos t$.



39. $y' = \cos(y - t)$

$$y' = c = \begin{cases} 0 \text{ when } y - t = -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dots \\ \qquad \text{or } y = t \pm (2n+1)\dfrac{\pi}{2} \\ 1 \text{ when } y - t = 0, 2\pi, \dots \\ \qquad \text{or } y = t \pm 2n\pi \\ -1 \text{ when } y - t = -\pi, \pi, 3\pi, \dots \\ \qquad \text{or } y = t \pm (2n+1)\pi \end{cases}$$



All these isoclines (dashed) have slope 1, with different y-intercepts.

The isoclines for *solution* slopes 1 are also solutions to the DE *and* act as oblique asymptotes for the other solutions between them (which, by uniqueness, do not cross. See Section 1.5).

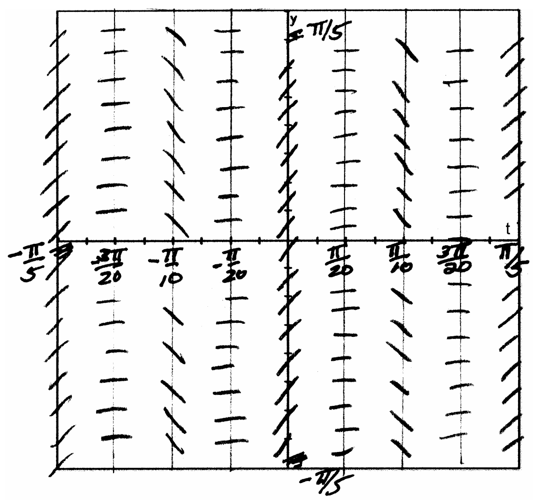
■ Periodicity

40. $y' = \cos 10t$

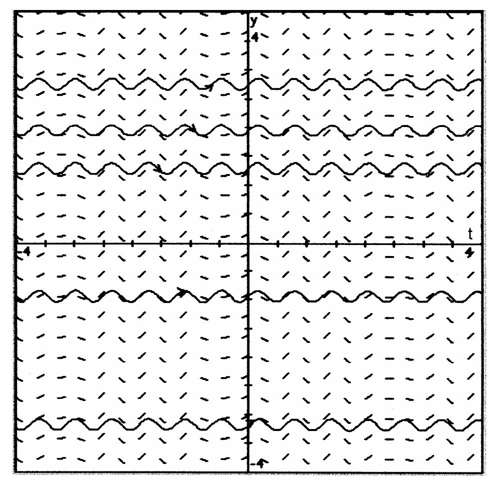
$$y' = c = \begin{cases} 0 \text{ when } 10t = \pm(2n+1)\left(\dfrac{\pi}{2}\right) \\ 1 \text{ when } 10t = \pm 2n\pi \\ -1 \text{ when } 10t = \pm(2n+1)\pi \end{cases}$$

y' is always between +1 and –1.

All solutions are periodic oscillations, with period $\dfrac{2\pi}{10}$.



Zooming in



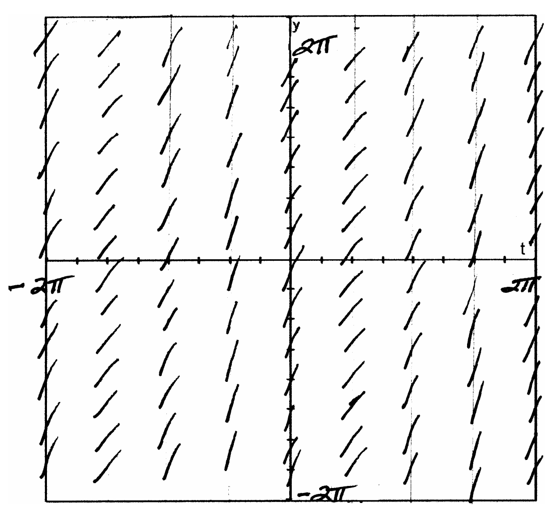
Zooming out

41. $y' = 2 - \sin t$

If $t = n\pi$, then $y' = 2$.

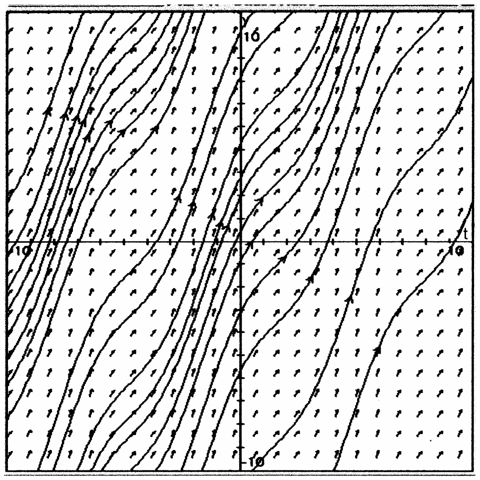
If $t = -\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \ldots$, then $y' = 1$.

All slopes are between 1 and 3.

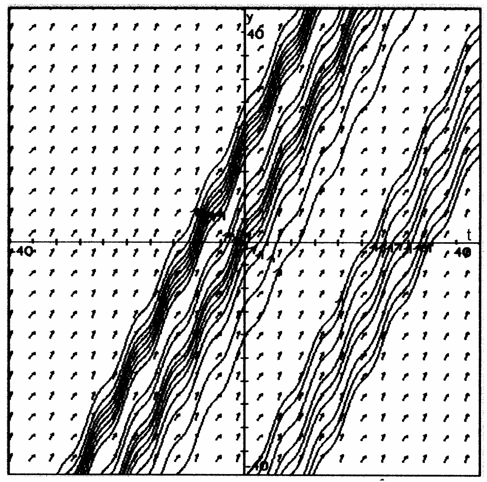


Although there is a periodic pattern to the direction field, the *solutions* are quite irregular and *not* periodic.

If you zoom out far enough, the oscillations of the solutions look somewhat more regular, but are always moving upward. See Figures.



Zooming out



Zooming further out

42. $y' = -\cos y$

If $y = \pm(2n+1)\dfrac{\pi}{2}$, then $y' = 0$ and these horizontal lines are equilibrium solutions.

For $y = \pm 2n\pi$, $y' = -1$

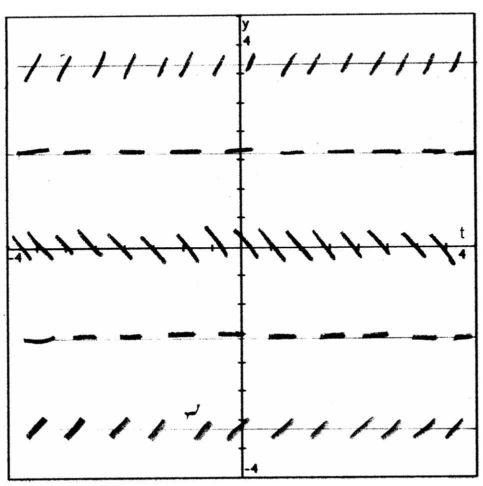
For $y = \pm(2n+1)\pi$, $y' = 1$.

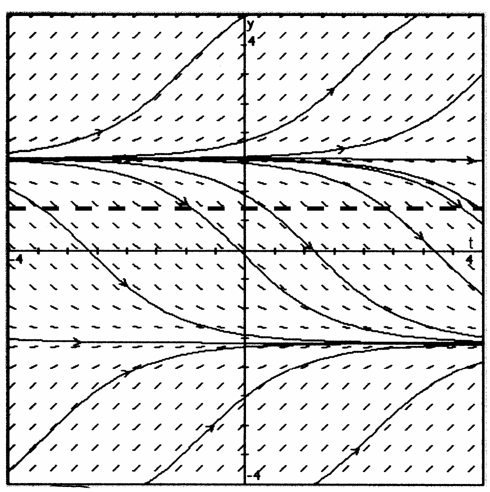
Slope y' is always between –1 and 1, and *solutions between the constant solutions cannot cross them, by uniqueness.*

To further check what happens in these cases we have added an isocline at $y = \dfrac{\pi}{4}$, where

$$y' = \cos\left(\frac{\pi}{4}\right) \approx -0.7.$$

Solutions are not periodic, but there is a periodicity to the direction field, in the vertical direction with period 2π. Furthermore, we observe that between every adjacent pair of constant solutions, the solutions are horizontal translates.





43. $y' = \cos 10t + 0.2$

For $10t = \pm(2n+1)\dfrac{\pi}{2}$

$$y' = 0.2,\ t \approx 0.157 \pm \frac{n\pi}{10}$$

For $10t = \pm 2n\pi$,

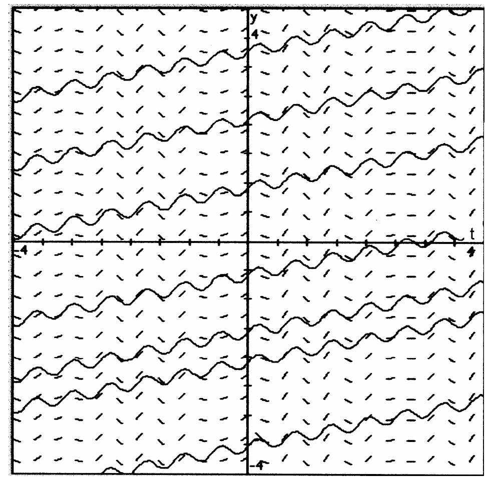
$$y' = 1.2,\ t \approx \pm\frac{2n\pi}{10}$$

For $10t = \pm(2n+1)\pi$

$$y' = -0.8,\ t \approx 0.314 \pm \frac{2n\pi}{10}$$

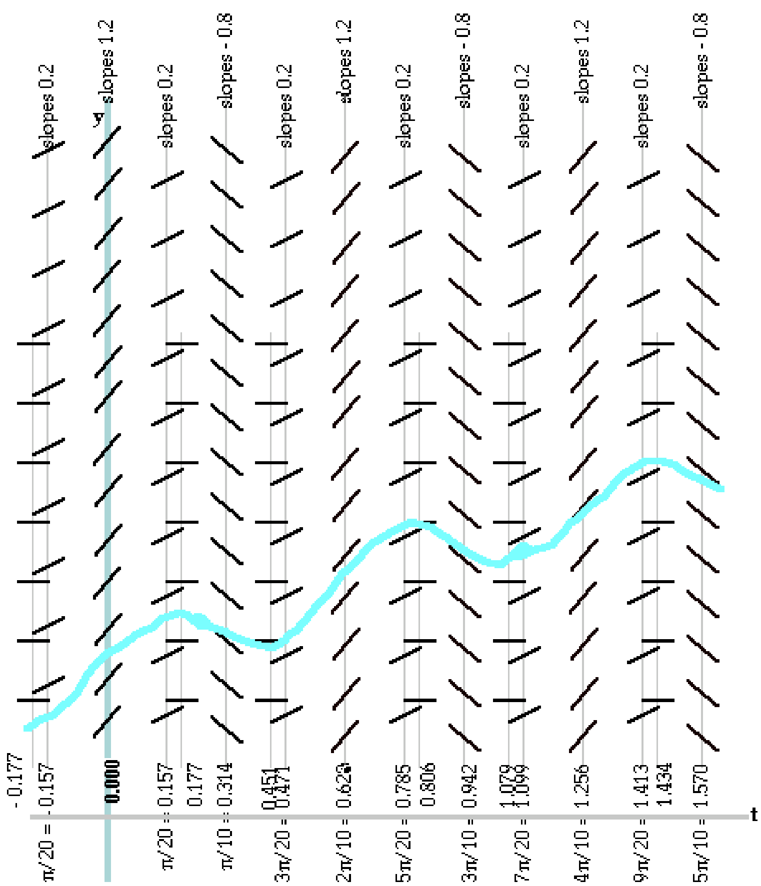
To get $y' = 0$ we must have $\cos 10t = -0.2$

Or $10t = \pm(1.77 + 2n\pi)$



Direction field and solutions over a larger scale.

The solutions oscillate in a periodic fashion, *but* at the same time they move ever upward. Hence they are *not* strictly periodic. Compare with Problem 40.



Direction field (augmented and improved in lower half), with rough sketch solution.

44. $y' = \cos(y - t)$

See Problem #39 for the direction field and sample solutions.

The solutions are not periodic, though there is a periodic (and diagonal) pattern to the overall direction field.

45. $y' = y(\cos t - y)$

Slopes are 0 whenever $y = \cos t$ or $y = 0$

Slopes are *negative* outside of both these isoclines;

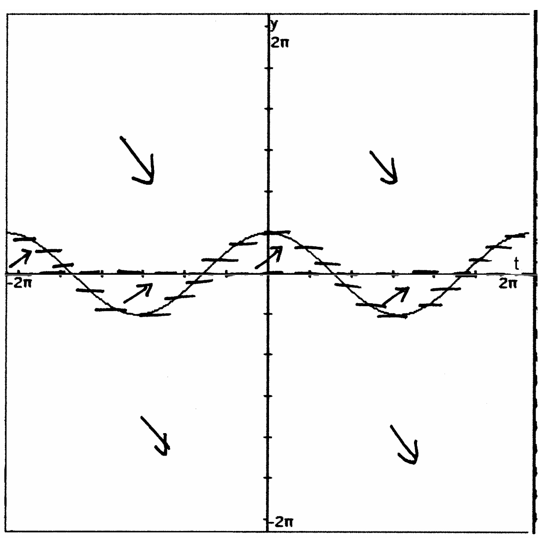
Slopes are *positive* in the regions trapped by the two isoclines.

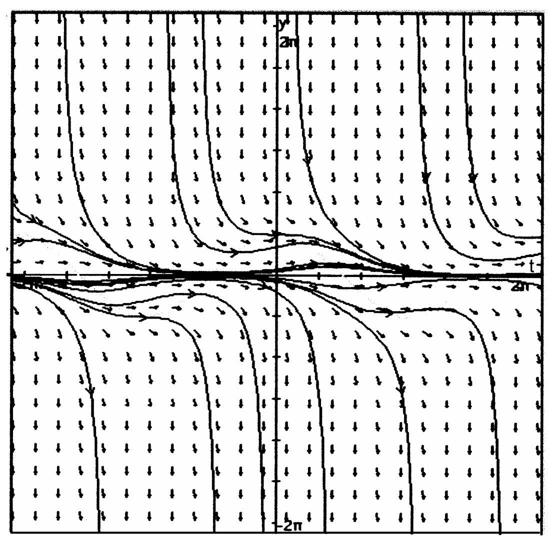
If you try to sketch a solution through this configuration, you will see it goes downward a lot more of the time than upward.

For $y > 0$ the solutions wiggle downward but never cross the horizontal axis—they get sent upward a bit first.

For $y < 0$ solutions eventually get out of the upward-flinging regions and go forever downward.

The solutions are *not* periodic, despite the periodic function in the DE.





46. $y' = \sin 2t + \cos t$

If $t = \pm 2n\pi$, then $y' = 0$.

If $t = \pm(2n+1)\dfrac{\pi}{2}$, then $y' = 0$.

If $t = \pm(2n + 1)\pi$, then $y' = -1$.

Isoclines are vertical lines, and solutions are vertical translates.

From this information it seems likely that solutions will oscillate with period 2π, rather like Problem 40. But beware—this is *not* the whole story. For $y' = \sin 2t + \cos t$, slopes will not remain between ± 1.

e.g.,

For $t = \dfrac{\pi}{4}, \dfrac{9\pi}{4}, \ldots,$ $y' \approx 1 + 0.7 = 1.7$.

For $t = \dfrac{3\pi}{4}, \dfrac{11\pi}{4}, \ldots,$ $y' \approx -1 - 0.7 = -1.7$.

For $t = \dfrac{5\pi}{4}, \dfrac{13\pi}{4}, \ldots,$ $y' \approx 1 - 0.7 = 0.3$

For $t = \dfrac{7\pi}{4}, \dfrac{15\pi}{4}, \ldots,$ $y' \approx -1 + 0.7 = -0.3$

The figures on the next page are crucial to seeing what is going on.

Adding these isoclines and slopes shows there are *more* wiggles in the solutions.

There are additional isoclines of zero slope where

$$\underbrace{\sin 2t}_{2\sin t\cos t} = -\cos t,$$

i.e., where $\sin t = -\dfrac{1}{2}$ and

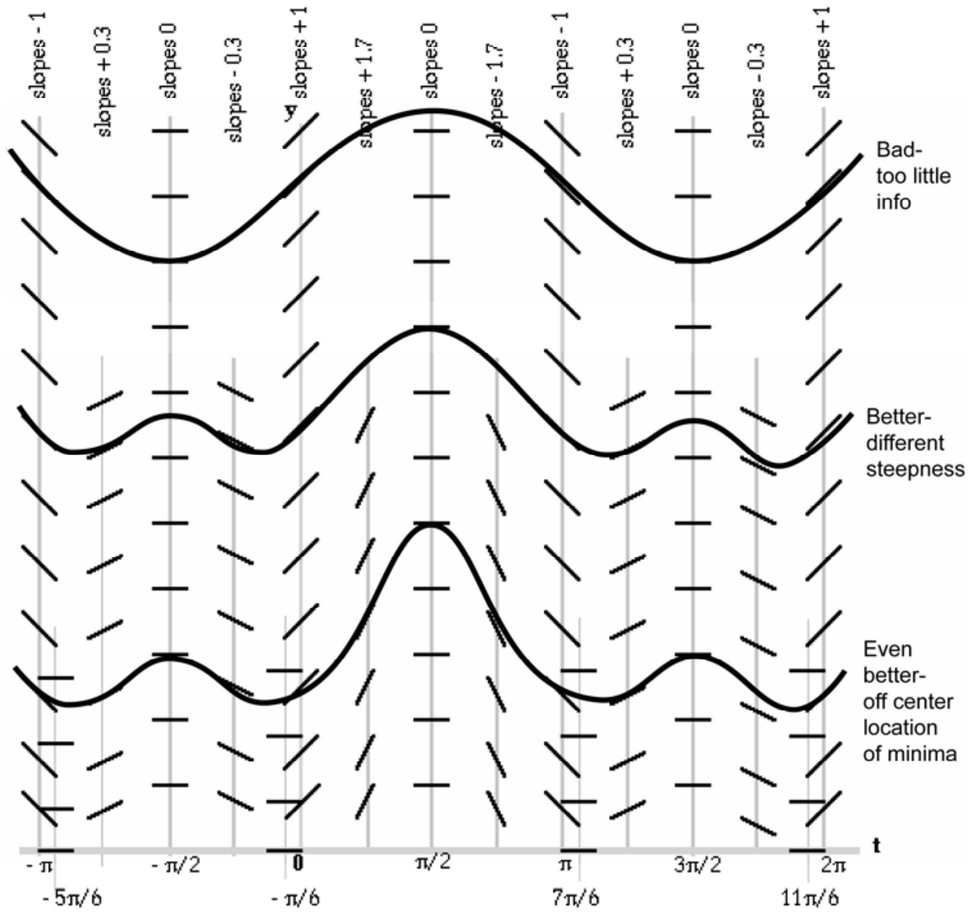
$$t = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \ldots$$

There is a symmetry to the slope marks about every vertical line where $t = \pm(2n+1)\dfrac{\pi}{2}$; these are some of the isoclines of zero slope.

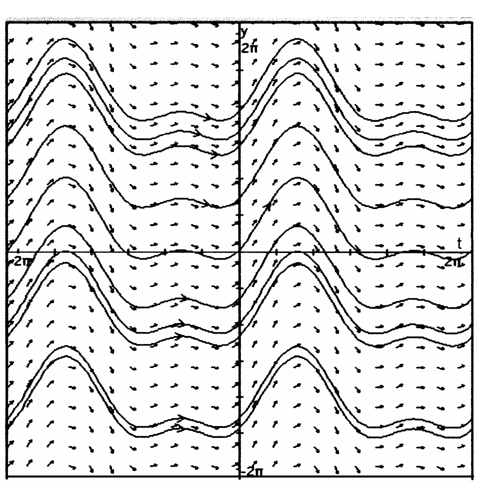
Solutions are periodic, with period 2π.

See figures on next page.

(**46.** continued)



Direction field, sketched with ever increasing detail as you move down the graph.



Direction field and solutions by computer.

■ **Symmetry**

47. $y' = y^2$

Note that y' depends only on y, so isoclines are horizontal lines.

Positive and negative values of y give the *same* slopes.

Hence the *slope values* are symmetric about the horizontal axis, but the resulting picture is *not*.

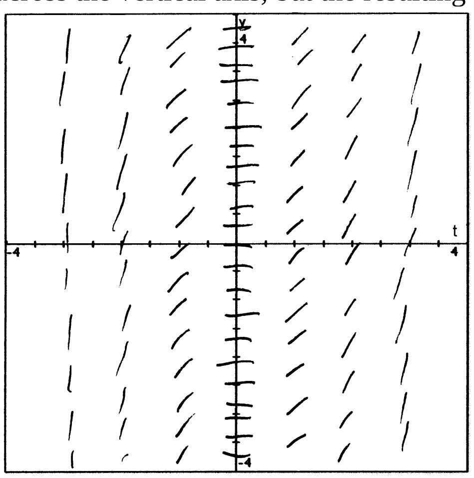
The figures are given with Problem 25 solutions.

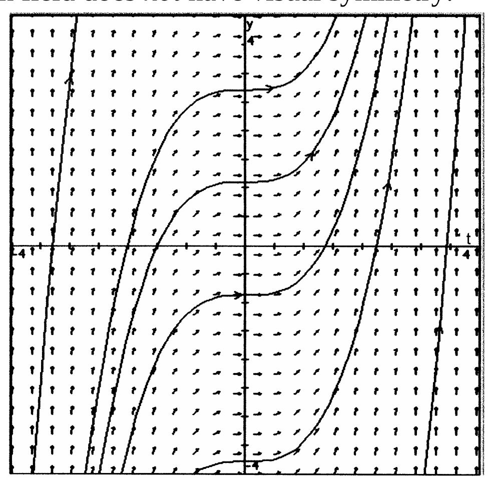
The only symmetry visible in the direction field is point symmetry, about the origin (or any point on the t-axis).

48. $y' = t^2$

Note that y' depends only on t, so isoclines are vertical lines.

Positive and negative values of t give the *same* slope, so the *slope values* are repeated symmetrically across the vertical axis, but the resulting direction field does *not* have visual symmetry.





The only symmetry visible in the direction field is point symmetry through the origin (or any point on the y-axis).

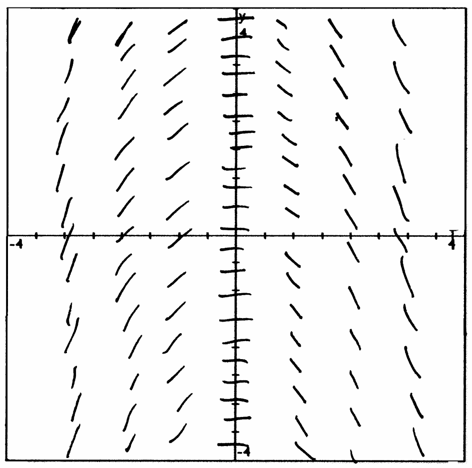
49. $y' = -t$

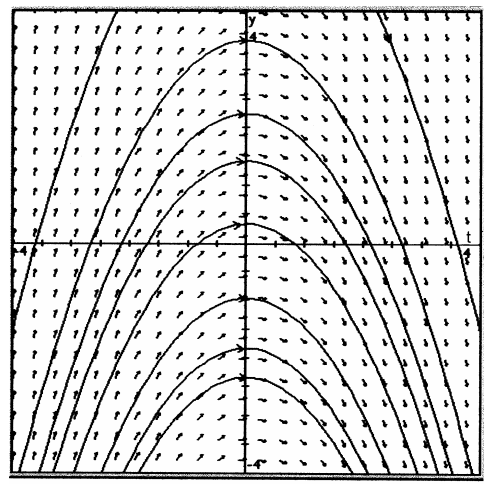
Note that y' depends only on t, so isoclines are vertical lines.

For $t > 0$, slopes are negative;

For $t < 0$, slopes are positive.

The result is pictorial symmetry of the vector field about the vertical axis.





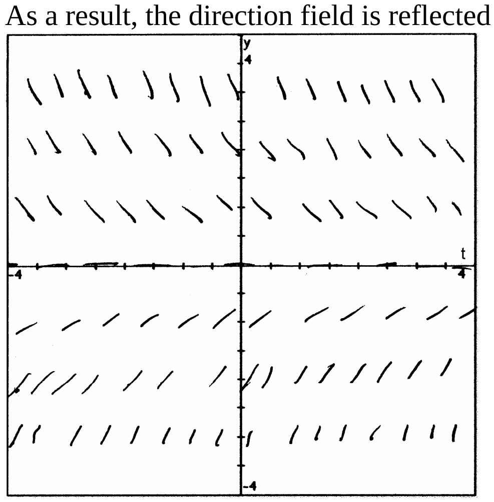
50. $y' = -y$

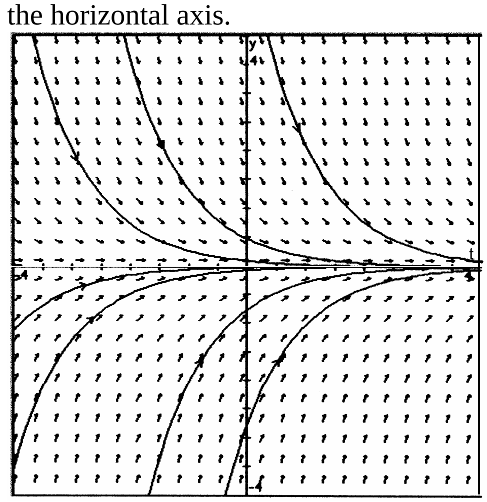
Note that y' depends only on y, so isoclines are horizontal lines.

For $y > 0$, slopes are negative.

For $y < 0$, slopes are positive.

As a result, the direction field is reflected across the horizontal axis.





51. $y' = \dfrac{1}{(t+1)^2}$

Note that y' depends only on t, so isoclines will be vertical lines.

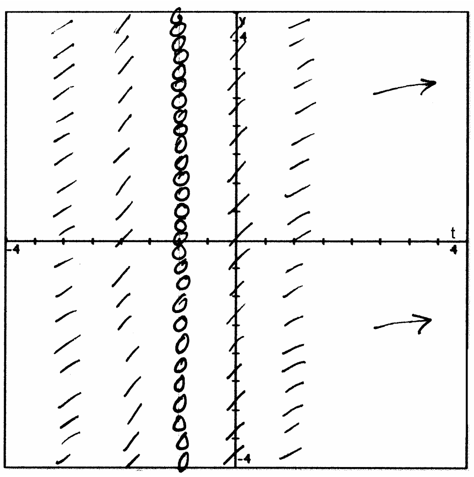
Slopes are always positive, so they will be *repeated*, not reflected, across $t = -1$, where the DE is not defined.

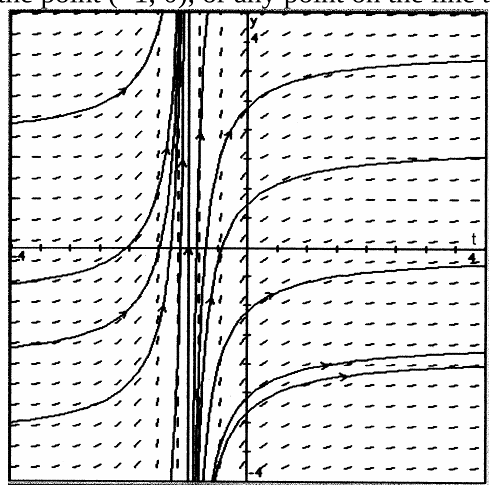
If $t = 0$ or -2, slope is 1.

If $t = 1$ or -3, slope is $\dfrac{1}{4}$.

If $t = 2$ or -4, slope is $\dfrac{1}{9}$.

The direction field has *point* symmetry through the point $(-1, 0)$, or any point on the line $t = -1$.





52. $y' = \dfrac{y^2}{t}$

Positive and negative values for y give the same slopes, $\dfrac{y^2}{t}$, so you can plot them for a single positive y-value and then *repeat* them for the negative of that y-value.

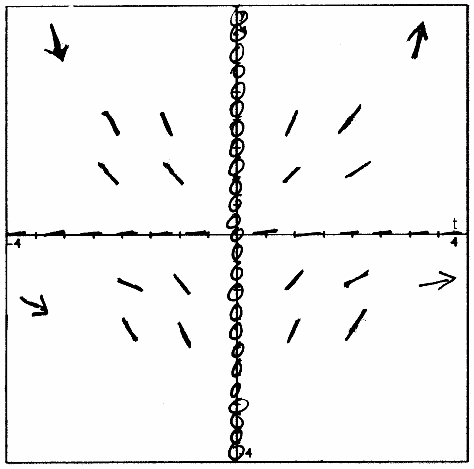
Note: Across the horizontal axis, this fact does *not* give symmetry to the direction field or solutions. However because the sign of t gives the sign of the slope, $\dfrac{y^2}{t}$, the result is a pictorial symmetry about the *vertical* axis. See figures on the next page.

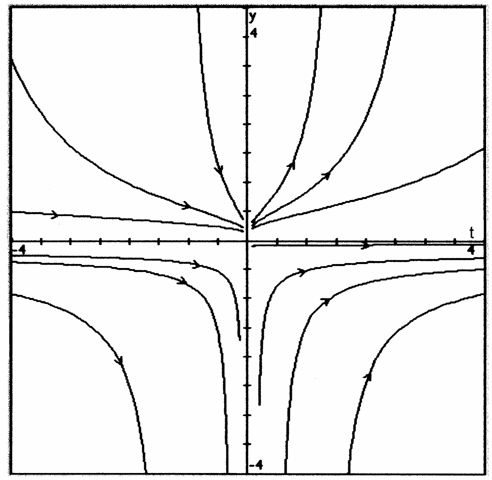
It is sufficient therefore to calculate slopes for the first quadrant only, that is, *reflect* them about the y-axis, *repeat* them about the t-axis.

If $y = 0$, $y' = 0$.

If $y = \pm 1$, $y' = \dfrac{1}{t}$.

If $y = \pm 2$ $y' = \dfrac{4}{t}$.





■ Second-Order Equations

53. (a) Direct substitution of y, y', and y'' into the differential equation reduces it to an identity.

(b) Direct computation

(c) Direct computation

(d) Substituting

$$y(t) = Ae^{2t} + Be^{-t}$$
$$y'(t) = 2Ae^{2t} - Be^{-t}$$

into the initial conditions gives

$$y(0) = A + B = 2$$
$$y'(0) = 2A - B = -5.$$

Solving these equations, gives $A = -1$, $B = 3$, so $y = -e^{-2t} + 3e^{-t}$.

■ Long-Term Behavior

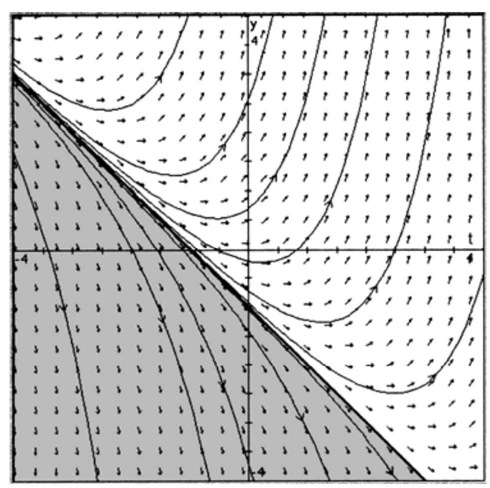
54. $y' = t + y$

(a) There are *no* constant solutions; zero slope requires $y = -t$, which is not constant.

(b) There are *no* points where the DE, or its solutions, are undefined.

(c) We see one straight line solution that appears to have slope $m = -1$ and y-intercept $b = -1$. Indeed, $y = -t - 1$ satisfies the DE.

(d) All solutions above $y = -t - 1$ are concave up; those below are concave down. This observation is confirmed by the sign of $y'' = 1 + y' = 1 + t + y$.



In shaded region, solutions are concave *down*.

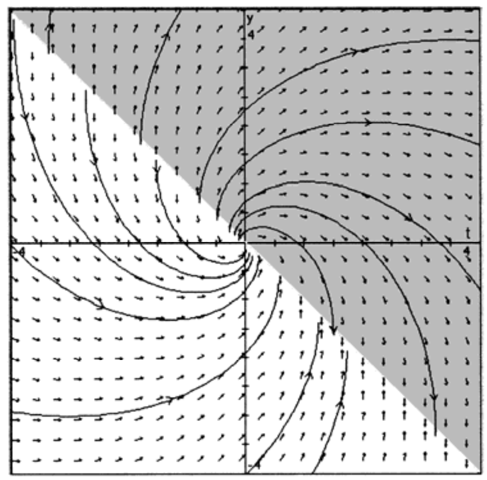
(e) As $t \to \infty$, solutions above $y = -t - 1$ approach ∞; those below approach $-\infty$.

(f) As $t \to -\infty$, going backward in time, *all* solutions are seen to emanate from ∞.

(g) The only asymptote, which is oblique, appears if we go backward in time—then all solutions are ever closer to $y = -t - 1$.

There are *no* periodic solutions.

55. $y' = \dfrac{y-t}{y+t}$



In shaded region, solutions are concave *down*.

(a) There are *no* constant solutions, but solutions will have zero slope along $y = t$.

(b) The DE is undefined along $y = -t$.

(c) There are *no* straight line solutions.

(d) $y'' = \dfrac{(y+t)(y'-1)-(y-t)(y'+1)}{(y+t)^2}$

Simplify using $y' - 1 = \dfrac{\overbrace{y-t-y-t}^{-2t}}{y+t}$

and $y' + 1 = \dfrac{\overbrace{y-t+y+t}^{2y}}{y+t}$, so that

$y'' = -2\dfrac{(t^2+y^2)}{(y+t)^3}$.

Hence y'' is $\begin{cases} \text{Never zero} \\ < 0 \text{ for } y+t>0, \text{ so solutions are concave down for } y > -t \\ > 0 \text{ for } y+t<0, \text{ so solutions are concave up for } y < -t \end{cases}$

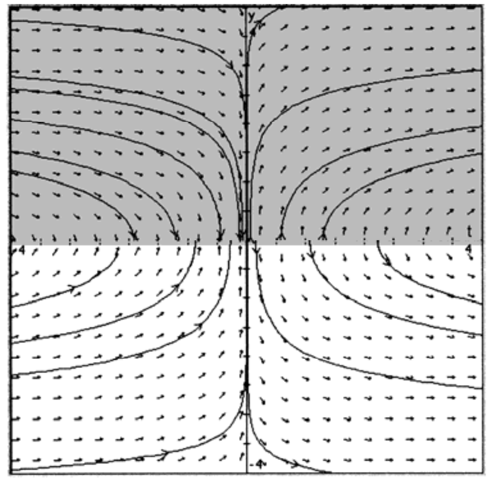
(e) As $t \to \infty$, all solutions approach $y = -t$.

(f) As $t \to -\infty$, we see that all solutions emanate from $y = -t$.

(g) All solutions become more vertical (at both ends) as they approach $y = -t$.

There are no periodic solutions.

56. $y' = \dfrac{1}{ty}$



In shaded region, solutions are concave *down*.

(a) There are *no* constant solutions, or even zero slopes, because $\dfrac{1}{ty}$ is never zero.

(b) The DE is undefined for $t = 0$ or for $y = 0$, *so solutions will not cross either axis*.

(c) There are *no* straight line solutions.

(d) Solutions will be concave down above the t-axis, concave up below the t-axis.

From $y' = \dfrac{1}{ty}$, we get

$$y'' = \frac{-1}{ty^2}y' - \frac{1}{t^2 y}.$$

This simplifies to

$y'' = -\dfrac{1}{t^2 y^3}\left(1 + y^2\right)$, which is never zero,

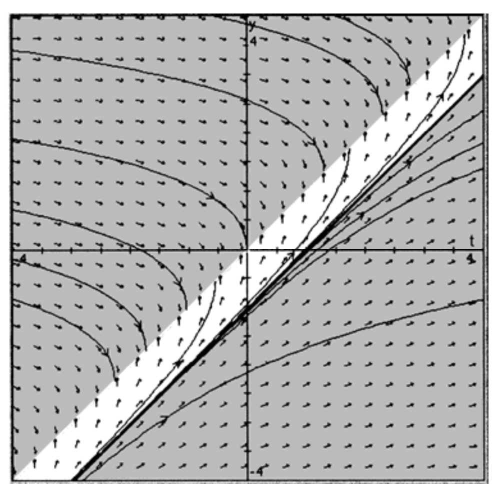
so there are no inflection points.

(e) As $t \to \infty$, solutions in upper quadrant $\to \infty$
solutions in the lower quadrant $\to -\infty$

(f) As $t \to -\infty$, we see that solutions in upper quadrant emanate from $+\infty$, those in lower quadrant emanate from $-\infty$.

(g) In the left and right half plane, solutions asymptotically approach vertical slopes as $y \to 0$. There are no periodic solutions.

57. $y' = \dfrac{1}{t - y}$



In shaded region, solutions are concave *down*.

(a) There are *no* constant solutions, nor even any point with zero slope.

(b) The DE is undefined along $y = t$.

(c) There appears to be one straight line solution with slope 1 and y-intercept -1; indeed $y = t - 1$ satisfies the DE.

$y' = 1$ when $y = t - 1$. Straight line solution

(d) $y'' = -\dfrac{(1-y')}{(t-y)^2} = \dfrac{y-(t-1)}{(t-y)^3}$

$y'' > 0$ when $y > t - 1$ and $y < t$ Solutions concave up

$$\left.\begin{aligned} y'' < 0 \text{ when } y < t-1 \text{ and } y > t \\ y > t-1 \text{ and } y > t \end{aligned}\right\} \text{Solutions concave down}$$

(e) As $t \to \infty$, solutions below $y = t - 1$ approach ∞;

solutions above $y = t - 1$ approach $y = t$ ever more vertically.

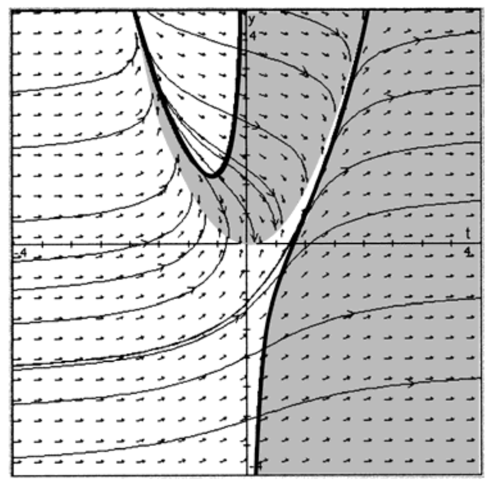
(f) As $t \to -\infty$, solutions above $y = t$ emanate from ∞;

solutions below $y = t$ emanate from $-\infty$.

(g) In *backwards* time the line $y = t - 1$ is an oblique asymptote.

There are no periodic solutions.

58. $y' = \dfrac{1}{t^2 - y}$



In shaded region, solutions are concave *down*. The DE is undefined on the boundary of the parabola. The dark curves are not solutions, but locus of inflection points

(a) There are *no* constant solutions.

(b) The DE is undefined along the parabola $y = t^2$, so solutions will not cross this locus.

(c) We see *no* straight line solutions.

(d) We see inflection points and changes in concavity, so we calculate

$y'' = -\dfrac{(2t - y')}{(t^2 - y)^2} = 0$ when $y' = 2t$

From DE $y' = \dfrac{1}{t^2 - y} = 2t$ when $y = t^2 - \dfrac{1}{2t}$, drawn as a thicker dashed curve with two branches.

Inside the parabola $y > t^2$, so $y' < 0$ and solutions are *decreasing*, concave *down* for solutions below the left branch of $y'' = 0$.

Outside the parabola $y < t^2$, $y' > 0$, solutions are *increasing*; and concave *down* below the right branch of $y'' = 0$.

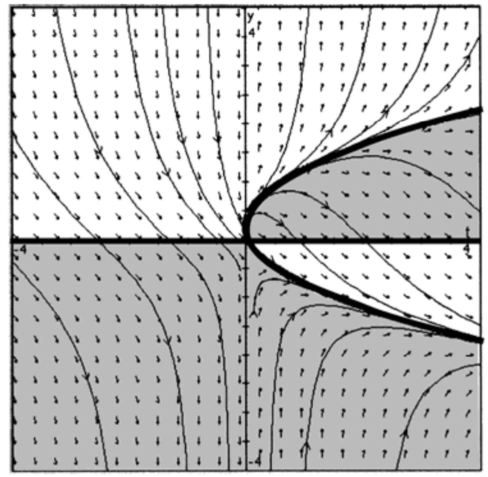
(e) As $t \to \infty$, slopes $\to 0$ and solutions $\to$ horizontal asymptotes.

(f) As $t \to -\infty$, solutions are seen to emanate from horizontal asymptotes.

(g) As solutions approach $y = t^2$, their slopes approach vertical.

There are no periodic solutions.

59. $y' = \dfrac{y^2}{t} - 1$



In shaded region, solutions are concave *down*. The horizontal axis is not a solution, just a locus of inflection points.

(a) There are *no* constant solutions.

(b) The DE is not defined for $t = 0$; solutions do not cross the y-axis.

(c) The only straight path in the direction field is along the y-axis, where $t = 0$. But the DE is not defined there, so there is no straight line solution.

(d) Concavity changes when

$$y'' = \frac{2yy't - y^2}{t^2} = \frac{y}{t^2}(2y^2 - y - 2t) = 0,$$

that is, when $y = 0$ or along the parabola

$$\left(t - \frac{1}{16}\right) = \left(y - \frac{1}{4}\right)^2$$

(obtained by solving the second factor of y'' for t and completing the square).

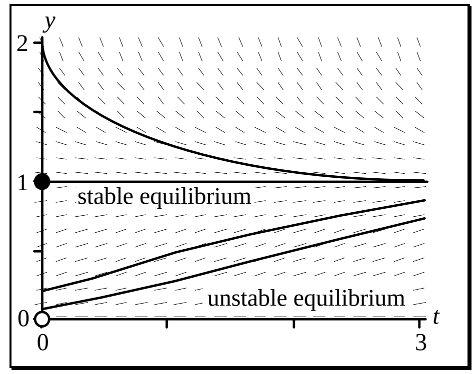
(e) As $t \to \infty$, most solutions approach $-\infty$. However in the first quadrant solutions above the parabola where $y'' = 0$ fly up toward $+\infty$. The parabola is composed of two solutions that act as a separator for behaviors of all the other solutions.

(f) In the left half plane solutions emanate from ∞.
In the right half plane, above the lower half of the parabola where $y'' = 0$, solutions seem to emanate from the upper y-intercept of the parabola; below the parabola they emanate from $-\infty$.

(g) The negative y-axis seems to be an asymptote for solutions in the left-half-plane, and in backward time for solutions in the lower right half plane.
There are *no* periodic solutions.

■ **Logistic Population Model**

60. We find the constant solutions by setting $y' = 0$ and solving for y. This gives $ky(1-y) = 0$, hence the constant solutions are $y(t) \equiv 0$, 1. Notice from the direction field or from the sign of the derivative that solutions starting at 0 or 1 remain at those values, and solutions starting between 0 and 1 increase asymptotically to 1, solutions starting larger than 1 decrease to 1 asymptotically. The following figure shows the direction field of $y' = y(1-y)$ and some sample solutions.



Logistic model

■ **Autonomy**

61. (a) Autonomous:

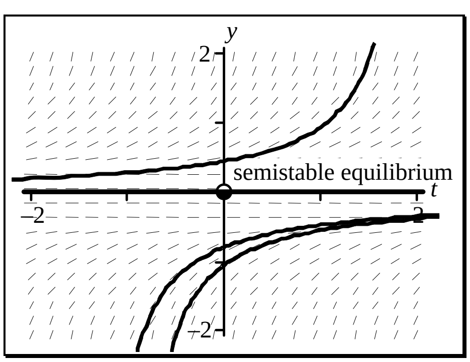
#9	$y' = 2y$
#13	$y' = 1 - y$
#14	$y' = y(y+1)$
#16	$y' = 1$
#17	$y' = y$
#32	$y' = -y$
#33	$y' = y^2$
#37	$y' = \cos y$

The others are nonautonomous.

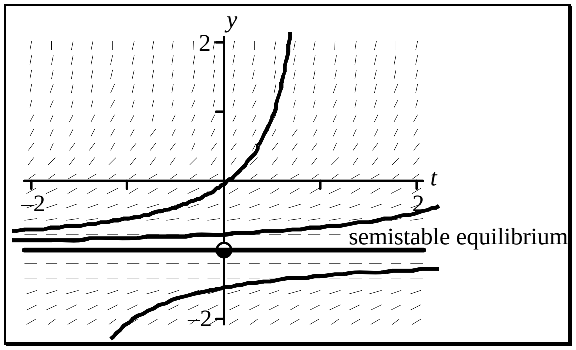
(b) Isoclines for autonomous equations consist of horizontal lines.

■ **Comparison**

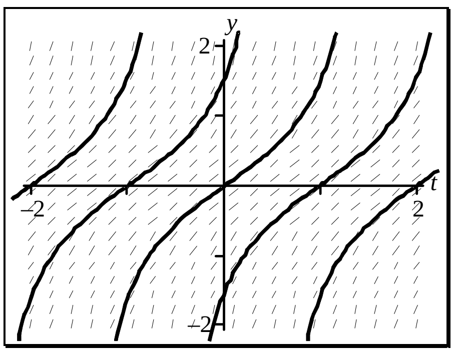
62. (i) $y' = y^2$



(ii) $y' = (y+1)^2$



(iii) $y' = y^2 + 1$



Equations (a) and (b) each have a constant solution that is unstable for higher values and stable for lower *y* values, but these equilibria occur at different levels. Equation (c) has no equilibrium at all.

All three DEs are autonomous, so within each graph solutions from left to right are always horizontal translates.

(a) For $y > 0$ we have

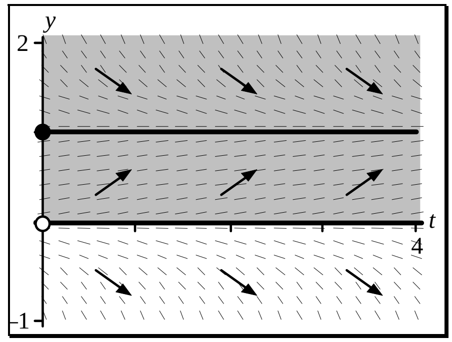
$$y^2 < y^2 + 1 < (y+1)^2.$$

For the three equations $y' = y^2$, $y' = y^2 + 1$, and $y' = (y+1)^2$, all with $y(0) = 1$; the solution of $y' = (y+1)^2$ will be the largest and the solution of $y' = y^2$ will be the smallest.

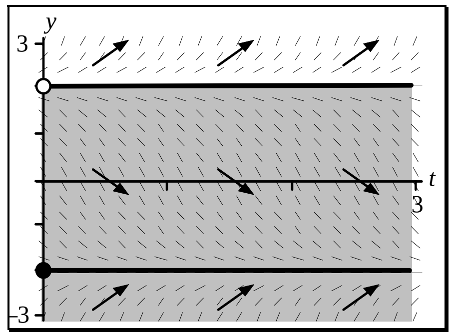
(b) Because $y(t) = \dfrac{1}{1-t}$ is a solution of the initial-value problem $y' = y^2$, $y(0) = 1$, which blows up at $t = 1$. We then know that the solution of $y' = y^2 + 1$, $y(0) = 1$ will blow up (approach infinity) somewhere between 0 and 1. When we solve this problem later using the method of separation of variables, we will find out *where* the solution blows up.

■ Coloring Basins

63. $y' = y(1-y)$. The constant solutions are found by setting $y' = 0$, giving $y(t) \equiv 0, 1$. Either by looking at the direction field or by analyzing the sign of the derivative, we conclude the constant solution $y(t) \equiv 1$ has a basin of attraction of $(0, \infty)$, and $y(t) \equiv 0$ has a basin attraction of the single value $\{0\}$. When the solutions have negative initial conditions, the solutions approach $-\infty$.

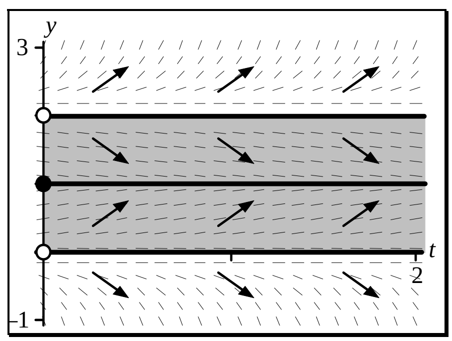


64. $y' = y^2 - 4$. The constant solutions are the (real) roots of $y^2 - 4 = 0$, or $y = \pm 2$. For $y > 2$, we have $y' > 0$. We, therefore, conclude solutions with initial conditions greater than 2 increase; for $-2 < y < 2$ we have $y' < 0$, hence solutions with initial conditions in this range decrease; and for $y < 0$, we have $y' > 0$, hence solutions with initial conditions in this interval increase.

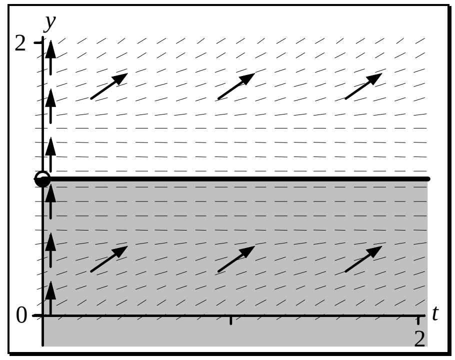


We can therefore, conclude that the constant solution $y = 2$ has a basin of attraction of the single value $\{2\}$, whereas the constant solution $y = -2$ has the basin of attraction of $(-\infty, 2)$

65. $y' = y(y-1)(y-2)$. Analyzing the sign of the derivative in each of the intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, \infty)$, we conclude that the constant solutions $y(t) \equiv 0, 1, 2$ have the following basins of attraction: $y(t) \equiv 0$ has the single point $\{0\}$ basin of attraction; $y(t) \equiv 1$ has the basin of attraction $(0, 2)$; and $y(t) \equiv 2$ has the single value $\{2\}$ basin of attraction.



66. $y' = (1-y)^2$. Because the derivative y' is always zero or positive, we conclude the constant solution $y(t) \equiv 1$ has basin of attraction the interval $(-\infty, 1]$.



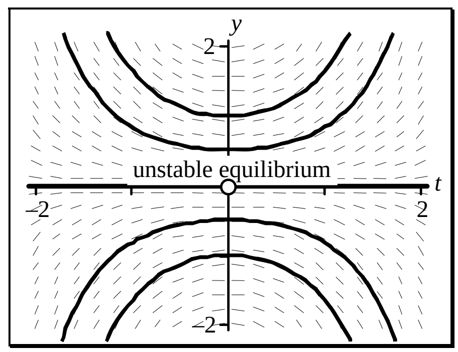
■ Computer or Calculator

The student can refer to Problems 69–73 as examples when working Problems 67, 68, and 74.

67. $y' = \dfrac{y}{2}$. Student Project

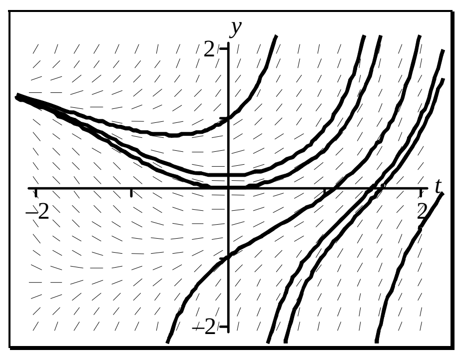
68. $y' = 2y + t$. Student Project

69. $y' = ty$. The direction field shows one constant solution $y(t) \equiv 0$, which is unstable (see figure). For negative t solutions approach zero slope, and for positive t solutions move away from zero slope.

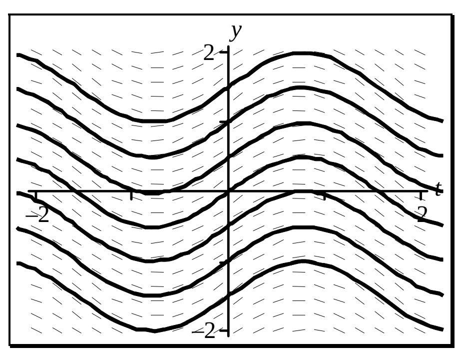


70. $y' = y^2 + t$. We see that eventually all solutions approach plus infinity. In *backwards* time most solutions approach the top part of this parabola.

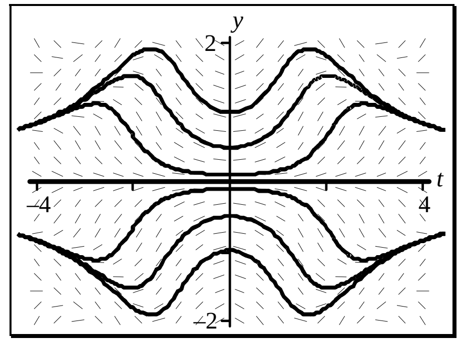
There are no constant or periodic solutions to this equation. You might also note that the isocline $y^2 + t = 0$ is a parabola sitting on its side for $t < 0$. In *backwards* time most solutions approach the top part of this parabola.



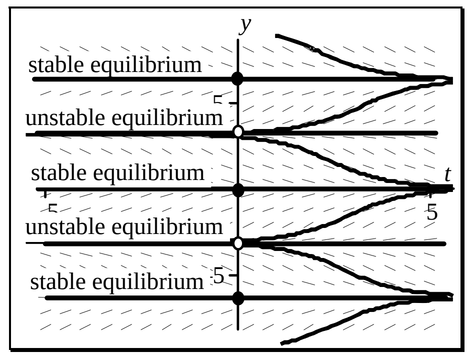
71. $y' = \cos 2t$. The direction field indicates that the equation has periodic solutions with the period roughly 3. This estimate is fairly accurate because $y(t) = \dfrac{1}{2}\sin 2t + c$ has period π.



72. $y' = \sin(ty)$. We have a constant solution $y(t) \equiv 0$ and there is a symmetry between solutions above and below the *t*-axis. Note: This equation does not have a closed form solution.



73. $y' = -\sin y$. We can see from the direction field that $y = 0,\ \pm\pi,\ \pm 2\pi,\ \ldots$ are constant solutions with $0,\ \pm 2\pi,\ \pm 4\pi,\ \ldots$ being stable and $\pm\pi,\ \pm 3\pi,\ \ldots$ unstable. The solutions between the equilibria have positive or negative slopes depending on the *y* interval. From left to right these solutions are horizontal translates.



74. $y' = 2y + t$. Student Project

■ **Suggested Journal Entry I**

75. Student Project

■ **Suggested Journal Entry II**

76. Student Project

1.3 Separation of Variables: Quantitative Analysis [redacted]

■ Separable or Not

1. $y' = 1 + y$. Separable; $\dfrac{dy}{1+y} = dt$; constant solution $y \equiv -1$.

2. $y' = y - y^3$. Separable; $\dfrac{dy}{y - y^3} = dt$; constant solutions $y(t) \equiv 0, \pm 1$.

3. $y' = \sin(t + y)$. Not separable; no constant solutions.

4. $y' = \ln(ty)$. Not separable; no constant solutions.

5. $y' = e^t e^y$. Separable; $e^{-y} dy = e^t dt$; no constant solutions.

6. $y' = \dfrac{y+1}{ty} + y$. Not separable; no constant solutions.

7. $y' = \dfrac{e^t e^y}{y+1}$. Separable; $e^{-y}(y+1)dy = e^t dt$; no constant solutions.

8. $y' = t\ln\left(y^{2t}\right) + t^2 = t^2(2\ln y + 1)$. Separable; $\dfrac{dy}{2\ln y + 1} = t^2 dt$; constant solution $y(t) \equiv e^{-1/2}$.

9. $y' = \dfrac{y}{t} + \dfrac{t}{y}$. Not separable; no constant solutions.

10. $y' = \dfrac{1+y^2}{t}$. Separable; $\dfrac{dy}{1+y^2} = dt/t$; no constant solution.

■ Solving by Separation

11. $y' = \dfrac{t^2}{y}$. Separating variables, we get $y\,dy = t^2 dt$. Integrating each side gives the implicit solution

$$\frac{1}{2}y^2 = \frac{1}{3}t^3 + c.$$

Solving for y yields branches so we leave the solution in implicit form.

12. $ty' = \sqrt{1 - y^2}$. The equilibrium solutions are $y = \pm 1$.

Separating variables, we get

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dt}{t}.$$

Integrating gives the implicit solution

$$\sin^{-1} y = \ln|t| + c.$$

Solving for y gives the explicit solution

$$y = \sin(\ln|t| + c).$$

13. $y' = \dfrac{t^2 + 7}{y^4 - 4y^3}$. Separating variables we get the equation

$$\left(y^4 - 4y^3\right)dy = \left(t^2 + 7\right)dt .$$

Integrating gives the implicit solution

$$\frac{1}{5}y^5 - y^4 = \frac{1}{3}t^3 + 7t + c .$$

We cannot find an explicit solution for y.

14. $ty' = 4y$. The equilibrium solution is $y = 0$.

Separating variables we get

$$\frac{dy}{y} = 4\frac{dt}{t} .$$

Integrating gives the implicit solution

$$\ln|y| = 4\ln|t| + c .$$

Solving for y gives the explicit solution

$$y = Ct^4$$

where C is an arbitrary constant.

15. $\dfrac{dy}{dt} = y\cos t$ $\quad y = 0$ is an equilibrium solution.

For $y \neq 0$, $\displaystyle\int\frac{dy}{y} = \int\cos t \; dt$

$\ln|y| = \sin t + c_1$

$e^{\ln|y|} = e^{\sin t}e^{c_1}$, so that $y = Ce^{\sin t}$, where $C = \pm e^{c_1}$.

16. $4t\, dy = (y^2 + ty^2)dt$ $\quad y(1) = 1$

$\displaystyle\int 4\frac{dy}{y^2} = \int\frac{1+t}{t}dt = \int\frac{1}{t} + 1 \; dt$

$-4y^{-1} = \ln|t| + t + C$

For $y(1) = 1$, we obtain $C = -5$, so that

$$y = \frac{-4}{\ln|t| + t - 5}$$

17. $y' = \dfrac{1-2t}{y}$, $y(1) = -2$. Separating variables gives

$$y\,dy = (1-2t)dt\,.$$

Integrating gives the implicit solution

$$\frac{1}{2}y^2 = t - t^2 + c\,.$$

Substituting in the initial condition $y(1) = -2$ gives $c = 2$. Hence, the implicit solution is given by

$$y^2 = 2t - 2t^2 + 4\,.$$

Solving for y we get

$$y(t) = -\sqrt{-2t^2 + 2t + 4}\,.$$

Note that we take the negative square root so the initial condition is satisfied.

18. $y' = y^2 - 4$, $y(0) = 0$. Separating variables gives

$$\frac{dy}{y^2 - 4} = dt\,.$$

Rewriting this expression as a partial fraction decomposition (see Appendix PF), we get

$$\left[\frac{1}{4(y-2)}1 - \frac{1}{4(y+2)}\right]dy = dt\,.$$

Integrating we get

$$\ln|y-2| - \ln|y+2| = 4t + c$$

or

$$\left|\frac{y-2}{y+2}\right| = e^c e^{4t}\,.$$

Hence, the implicit solution is

$$\frac{y-2}{y+2} = \pm e^c e^{4t} = ke^{4t}$$

where k is an arbitrary constant. Solving for y, we get the general solution

$$y(t) = \frac{2\left(1 + ke^{4t}\right)}{1 - ke^{4t}}.$$

Substituting in the initial condition $y(0) = 0$ gives $k = -1$.

19. $y' = \dfrac{2t}{1+2y}$, $y(2) = 0$. Separating variables

$$(1+2y)dy = 2t\,dt.$$

Integrating gives the implicit solution

$$y + y^2 = t^2 + c.$$

Substituting in the initial condition $y(2) = 0$ gives $c = -4$. Solving for y the preceding quadratic equation in y we get

$$y = \frac{-1+\sqrt{1+4(t^2-4)}}{2}.$$

20. $y' = -\dfrac{1+y^2}{1+t^2}$, $y(0) = -1$. Separating variables, we get the equation

$$\frac{dy}{1+y^2} = -\frac{dt}{1+t^2}.$$

Integrating gives

$$\tan^{-1} y = -\tan^{-1} t + c.$$

Substituting in the initial condition $y(0) = -1$ gives $c = \tan^{-1}(-1) = -\dfrac{\pi}{4}$. Solving for y gives

$$y = \tan\left(-\tan^{-1} t - \frac{\pi}{4}\right).$$

■ Integration by Parts

21. $y' = \left(\cos^2 y\right)\ln t$. The equilibrium solutions are $y = (2n+1)\dfrac{\pi}{2}$.

Separating variables we get

$$\frac{dy}{\cos^2 y} = \ln t dt.$$

Integrating, we find

$$\begin{aligned} \int \frac{dy}{\cos^2 y} &= \int \ln t\,dt + c \\ \int \sec^2 y\,dy &= \int \ln t\,dt + c \\ \tan y &= t\ln t - t + c \\ y &= \tan^{-1}(t\ln t - t + c). \end{aligned}$$

22. $y' = (t^2 - 5)\cos 2t$. Separating variables we get

$$dy = (t^2 - 5)\cos 2t\,dt.$$

Integrating, we find

$$\begin{aligned} y &= \int (t^2 - 5)\cos 2t\,dt + c \\ &= \int t^2 \cos 2t\,dt - 5\int \cos 2t\,dt + c \\ &= \frac{1}{4}(2t^2 - 1)\sin 2t + \frac{1}{2}t\cos 2t - \frac{5}{2}\sin 2t + c. \end{aligned}$$

23. $y' = t^2 e^{y+2t}$. Separating variables we get

$$\frac{dy}{e^y} = t^2 e^{2t} dt.$$

Integrating, we find

$$\begin{aligned} \int e^{-y} dy &= \int t^2 e^{2t} dt + c \\ -e^{-y} &= \frac{1}{2}(t^2 - t)e^{2t} + \frac{1}{4}e^{2t} + c \\ e^{-y} &= -\left[\frac{1}{2}(t^2 - t)e^{2t} + \frac{1}{4}e^{2t} + c\right]. \end{aligned}$$

Solving for y, we get

$$y = -\ln\left[\frac{1}{2}(t - t^2)e^{2t} - \frac{1}{4}e^{2t} - c\right].$$

24. $y' = t\,ye^{-t}$. The equilibrium solution is $y = 0$.

Separating variables we get

$$\frac{dy}{y} = te^{-t} dt.$$

Integrating, we find

$$\begin{aligned} \int \frac{dy}{y} &= \int te^{-t} dt + c \\ \ln|y| &= -te^{-t} - e^{-t} + c \\ y &= Qe^{-(t+1)e^{-t}}. \end{aligned}$$

■ Equilibria and Direction Fields

25. (C) **26.** (B) **27.** (E) **28.** (F) **29.** (A) **30.** (D)

■ Finding the Nonequilibrium Solutions

31. $y' = 1 - y^2$

We note first that $y = \pm 1$ are equilibrium solutions. To find the nonconstant solutions we divide by $1 - y^2$ and rewrite the equation in differential form as

$$\frac{dy}{1 - y^2} = dt.$$

By a partial fraction decomposition (see Appendix PF), we have

$$\frac{dy}{(1 - y)(1 + y)} = \frac{dy}{2(1 - y)} + \frac{dy}{2(1 + y)} = dt.$$

Integrating, we find

$$-\frac{1}{2}\ln|1 - y| + \frac{1}{2}\ln|1 + y| = t + c$$

where c is any constant. Simplifying, we get

$$\begin{aligned} -\ln|1 - y| + \ln|1 + y| &= 2t + 2c \\ \ln\left|\frac{1 + y}{1 - y}\right| &= 2t + 2c \\ \left|\frac{(1 + y)}{(1 - y)}\right| &= ke^{2t} \end{aligned}$$

where k is any nonzero real constant. If we now solve for y, we find

$$y = \frac{ke^{2t} - 1}{ke^{2t} + 1}.$$

32. $y' = 2y - y^2$

We note first that $y = 0$, 2 are equilibrium solutions. To find the nonconstant solutions, we divide by $2y - y^2$ and rewrite the equation in differential form as

$$\frac{dy}{y(2 - y)} = dt.$$

By a partial fraction decomposition (see Appendix PF),

$$\frac{dy}{y(2 - y)} = \frac{dy}{2y} + \frac{dy}{2(2 - y)} = dt.$$

Integrating, we find

$$\frac{1}{2}\ln|y| - \frac{1}{2}\ln|2 - y| = t + c$$

where c is any real constant. Simplifying, we get

$$\ln|y| - \ln|2-y| = 2t + 2c$$
$$\ln\left|\frac{y}{2-y}\right| = 2t + 2c$$
$$\left|y(2-y)\right| = Ce^{2t}$$

where C is any positive constant.

$$\frac{y}{2-y} = ke^{2t}$$

where k is any nonzero real constant. If we solve for y, we get

$$y = \frac{2ke^{2t}}{1+ke^{2t}}.$$

33. $y' = y(y-1)(y+1)$

We note first that $y = 0,\ \pm 1$ are equilibrium solutions. To find the nonconstant solutions, we divide by $y(y-1)(y+1)$ and rewrite the equation in differential form as

$$\frac{dy}{y(y-1)(y+1)} = dt\,.$$

By finding a partial fraction decomposition, (see Appendix PF)

$$\frac{dy}{y(y-1)(y+1)} = -\frac{dy}{y} + \frac{dy}{2(y-1)} + \frac{dy}{2(y+1)} = dt\,.$$

Integrating, we find

$$-\ln|y| + \frac{1}{2}\ln|y-1| + \frac{1}{2}|y+1| = t + c$$
$$-2\ln|y| + \ln|y-1| + \ln|y+1| = 2t + 2c$$

or

$$\ln\left|\frac{(y-1)(y+1)}{y^2}\right| = 2t + 2c$$
$$\frac{(y-1)(y+1)}{y^2} = ke^{2t}\,.$$

Multiplying each side of the above equation by y^2 gives a quadratic equation in y, which can be solved, getting

$$y = \pm\sqrt{\frac{1}{(1+ke^{2t})}}\,.$$

Initial conditions will tell which branch of this solution would be used.

34. $y' = (y-1)^2$

We note that $y = 1$ is a constant solution. Seeking nonconstant solutions, we divide by $(y-1)^2$ getting $\dfrac{dy}{(y-1)^2} = dt$. This can be integrated to get

$$-\frac{1}{y-1} = t + c$$

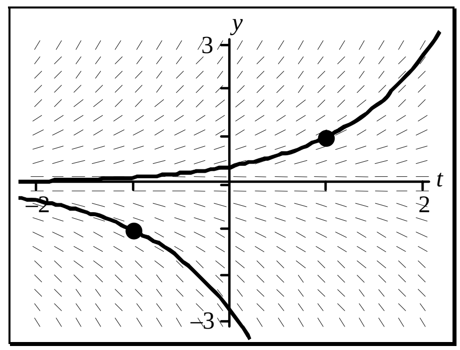
$$y - 1 = \frac{1}{-t+c}$$

$$y = 1 + \frac{1}{-t+c}.$$

■ Help from Technology

35. $y' = y,\ y(1) = 1,\ y(-1) = -1$

The solution of $y' = y,\ y(1) = 1$ is $y = e^{t-1}$. The solution of $y' = y,\ y(-1) = -1$ is $y = -e^{t+1}$. These solutions are shown in the figure.



36. $y' = \cos t,\ y(1) = 1,\ y(-1) = -1$

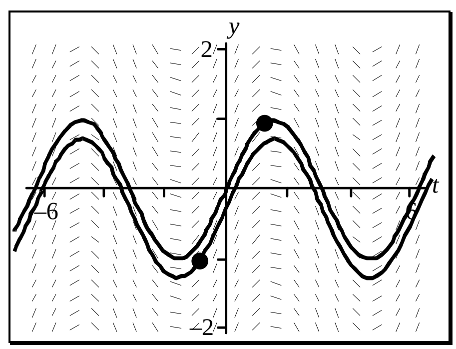
The solution of the initial-value problem

$$y' = \cos t,\ y(1) = 1$$

is $y(t) = \sin t + 1 - \sin(1)$. The solution of

$$y' = \cos t,\ y(-1) = -1$$

is $y = \sin t - 1 + \sin(-1)$. The solutions are shown in the figure.



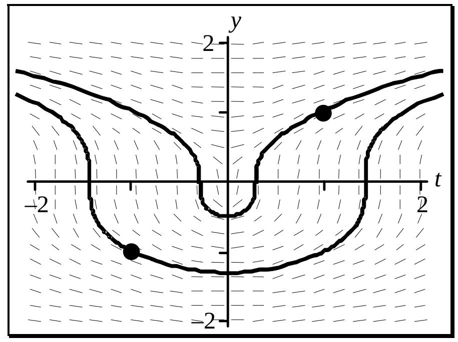
37. $\dfrac{dy}{dt} = \dfrac{t}{y^2\sqrt{1+t^2}}$, $y(1) = 1$, $y(-1) = -1$

Separating variables and integrating we find the implicit solution

$$\int y^2 dy = \int \frac{t}{\sqrt{1+t^2}}\, dt + c$$

or

$$\frac{1}{3}y^3 = \sqrt{1+t^2} + c.$$



$$y' = \frac{t}{y^2\sqrt{1+t^2}}$$

Subsituting $y(1) = 1$, we find $c = \frac{1}{3} - \sqrt{2}$. For $y(-1) = -1$ we find $c = -\frac{1}{3} - \sqrt{2}$. These two curves are shown in the figure.

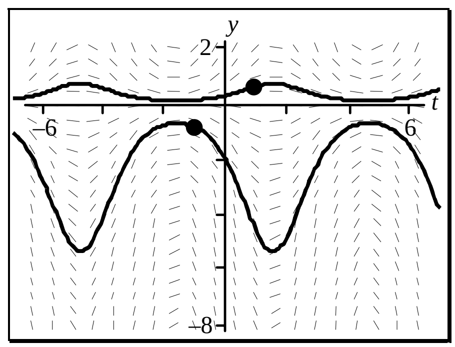
38. $y' = y\cos t$, $y(1) = 1$, $y(-1) = -1$

Separating variables we get

$$\frac{dy}{y} = \cos t\, dt.$$

Integrating, we find the implicit solution

$$\ln|y| = \sin t + c.$$



With $y(1) = 1$, we find $c = -\sin(1)$. With $y(-1) = -1$, we find $c = \sin(1)$. These two implicit solution curves are shown imposed on the direction field (see figure).

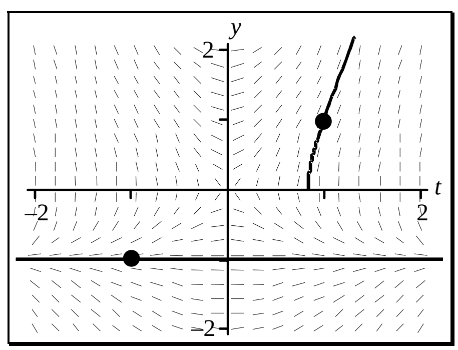
39. $y' = \dfrac{2t(y+1)}{y}$, $y(1) = 1$, $y(-1) = -1$

Separating variables and assuming $y \neq -1$, we find

$$\frac{y}{y+1}dy = 2t\, dt$$

or

$$\int \frac{y}{y+1}dy = \int 2t\, dt + c.$$



$$y' = \frac{2t(y+1)}{y}$$

Integrating, we find the implicit solution

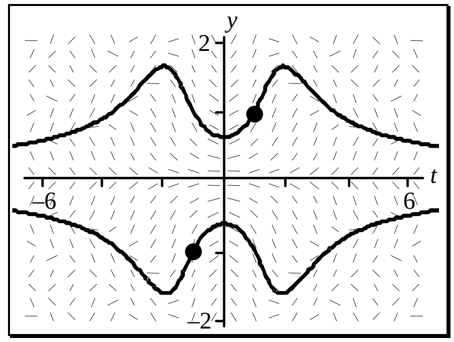
$$y - \ln|y+1| = t^2 + c.$$

For $y(1) = 1$, we get $1 - \ln 2 = 1 + c$ or $c = -\ln 2$. For $y(-1) = -1$ we can see even more easily that $y \equiv -1$ is the solution. These two solutions are plotted on the direction field (see figure). Note that

the implicit solution involves branching. The initial condition $y(1)=1$ lies on the upper branch, and the solution through that point does not cross the t-axis.

40. $y'=\sin(ty)$, $y(1)=1$, $y(-1)=-1$

This equation is not separable and has no closed form solution. However, we can draw its direction field along with the requested solutions (see figure).



■ Making Equations Separable

41. Given

$$y'=\frac{y+t}{t}=1+\frac{y}{t},$$

we let $v=\dfrac{y}{t}$ and get the separable equation $v+t\dfrac{dv}{dt}=1+v$. Separating variables gives

$$\frac{dt}{t}=dv\,.$$

Integrating gives

$$v=\ln|t|+c$$

and

$$y=t\ln|t|+ct\,.$$

42. Letting $v=\dfrac{y}{t}$, we write

$$y'=\frac{y^2+t^2}{yt}=\frac{y}{t}+\frac{t}{y}=v+\frac{1}{v}\,.$$

But $y=tv$ so $y'=v+tv'$. Hence, we have

$$v+tv'=v+\frac{1}{v}$$

or

$$tv'=\frac{1}{v}$$

or

$$v\,dv=\frac{dt}{t}\,.$$

Integrating gives the implicit solution

$$\frac{1}{2}v^2 = \ln|t| + c$$

or

$$v = \pm\sqrt{2\ln|t| + c} .$$

But $v = \frac{y}{t}$, so

$$y = \pm t\sqrt{2\ln|t| + c} .$$

The initial condition $y(1) = -2$ requires the negative square root and gives $c = 4$. Hence,

$$y(t) = -t\sqrt{2\ln|t| + 4} .$$

43. Given

$$y' = \frac{2y^4 + t^4}{ty^3} = \frac{2y}{t} + \frac{t^3}{y^3} = 2v + \frac{1}{v^3} .$$

with the new variable $v = \frac{y}{t}$. Using $y' = v + tv'$ and separating variables gives

$$\frac{dt}{t} = \frac{dv}{\frac{v^4+1}{v^3}} = v^3\frac{dv}{v^4 + 1} .$$

Integrating gives the solution

$$\ln|t| = \frac{1}{4}\ln\left(v^4 + 1\right) + c$$

or

$$\ln|t| = \frac{1}{4}\ln\left[\left(\frac{y}{t}\right)^4 + 1\right] + c.$$

44. Given

$$y' = \frac{y^2 + ty + t^2}{t^2} = \frac{y^2}{t^2} + \frac{y}{t} + 1 = v^2 + v + 1$$

with the new variable $v = \frac{y}{t}$. Using $y' = v + tv'$ and separating variables, we get

$$\frac{dv}{v^2 + 1} = \frac{dt}{t} .$$

Integrating gives the implicit solution

$$\ln|t| = \tan^{-1} v + c .$$

Solving for v gives $v = \tan\left(\ln|t| + c\right)$. Hence, we have the explicit solution

$$y = t\tan(\ln|t| + c).$$

■ Another Conversion to Separable Equations

45. $y' = (y+t)^2$ Let $u = y + t$. Then

$$\frac{du}{dt} = \frac{dy}{dt} + 1 = u^2 + 1, \text{ and } \int \frac{du}{u^2+1} = \int dt, \text{ so}$$

$$\begin{aligned} \tan^{-1} u &= t + c \\ u &= \tan(t+c) \\ y + t &= \tan(t+c) \text{ so } y = \tan(t+c) - t \end{aligned}$$

46. $\frac{dy}{dt} = e^{t+y-1} - 1$ Let $u = t + y - 1$. Then

$$\frac{du}{dt} = 1 + \frac{dy}{dt} = 1 + e^u - 1, \text{ and } \int e^{-u}du = \int dt, \text{ so}$$

$$-e^{-u} + c = t, \quad \text{or} \quad t + e^{-t-y+1} = c.$$

Thus, $-t - y + 1 = \ln|c-t|$, and $y = 1 - t - \ln|c-t|$.

■ Autonomous Equations

47. (a) Problems 1, 2 and 18 are autonomous:

$$\begin{aligned} &\#1 \quad y' = 1 + y \\ &\#2 \quad y' = y - y^3 \\ &\#18 \quad y' = y^2 - 4 \end{aligned}$$

All the others are nonautonomous.

(b) The isoclines of an autonomous equation are horizontal lines (i.e., if you follow along a horizontal line $y = k$ in the ty plane, the slopes of the line elements do not change). Another way to say this is that solutions for $y(t)$ through any y all have the same slope.

■ Orthogonal Families

48. (a) Starting with $f(x, y) = c$, we differentiate implicitly getting the equation

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Solving for $y' = \frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

These slopes are the slopes of the tangent lines.

(b) Taking the negative reciprocal of the slopes of the tangents, the orthogonal curves satisfy

$$\frac{dy}{dx} = \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}.$$

(c) Given $f(x, y) = x^2 + y^2$, we have

$$\frac{\partial f}{\partial y} = 2y \text{ and } \frac{\partial f}{\partial x} = 2x,$$

so our equation is $\frac{dy}{dx} = \frac{y}{x}$. Hence, from part (b) the orthogonal trajectories satisfy the differential equation

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{y}{x},$$

which is a separable equation having solution $y = kx$.

■ More Orthogonal Trajectories

49. For the family $y = cx^2$ we have $f(x, y) = \frac{y}{x^2}$ so

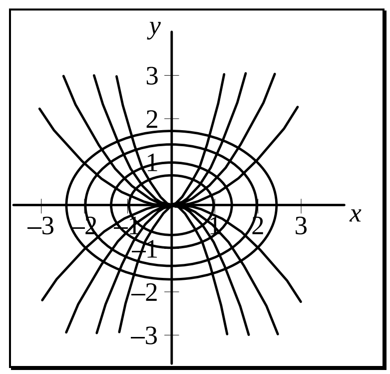
$$f_x = -\frac{2y}{x^3},\ f_y = \frac{1}{x^2},$$

and the orthogonal trajectories satisfy

$$\frac{dy}{dx} = \frac{f_y}{f_x} = -\frac{x}{2y}$$

or

$$2y\,dy = -x\,dx.$$



Orthogonal trajectories

Integrating, we have

$$y^2 = -\frac{1}{2}x^2 + K$$

or

$$x^2 + 2y^2 = C.$$

Hence, this last equation gives a family of ellipses that are all orthogonal to members of the family $y = cx^2$. Graphs of the orthogonal families are shown in the figure.

50. For the family $y = \dfrac{c}{x^2}$ we have $f(x, y) = x^2 y$ so

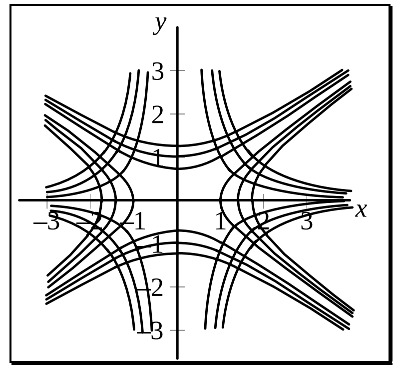
$$f_x = 2xy, \; f_y = x^2$$

and the orthogonal trajectories satisfy

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{x}{2y}$$

or, in differential form, $2y\,dy = x\,dx$. Integrating, we have

$$y^2 = \frac{1}{2}x^2 + C \text{ or } 2y^2 - x^2 = K.$$



Orthogonal trajectories

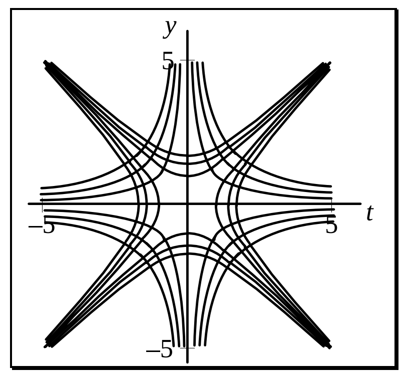
Hence, the preceding equations give a family of hyperbolas that are orthogonal to the original family of hyperbolas $y = \dfrac{c}{x^2}$. Graphs of the two orthogonal families of hyperbolas are shown.

51. $xy = c$. Here $f(x, y) = xy$ so $f_x = y$, $f_y = x$. The orthogonal trajectories satisfy

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{x}{y}$$

or, in differential form, $y\,dy = x\,dx$. Integrating, we have the solution

$$y^2 - x^2 = C.$$

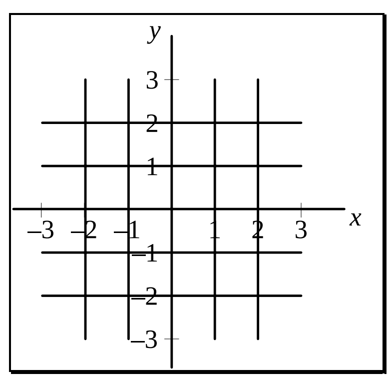


Orthogonal hyperbolas

Hence, the preceding family of hyperbolas are orthogonal to the hyperbolas $xy = c$. Graphs of the orthogonal families are shown.

■ Calculator or Computer

52. $y = c$. We know the orthogonal trajectories of this family of horizontal lines is the family of vertical lines $x = C$ (see figure).



Orthogonal trajectories

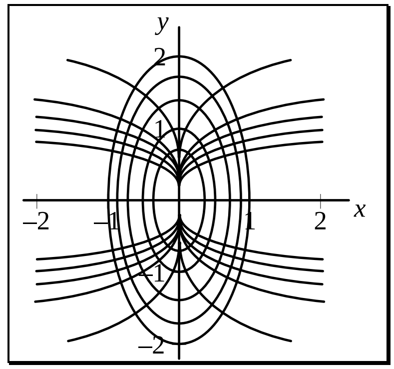
53. $4x^2 + y^2 = c$. Here

$$f(x, y) = 4x^2 + y^2$$

and $f_x = 8x$, $f_y = 2y$, so the orthogonal trajectories satisfy

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{2y}{8x} = \frac{y}{4x}$$

or $\frac{4dy}{y} = \frac{dx}{x}$, which has the implicit solution the family $y^4 = Cx$ where C is any constant different from zero. These orthogonal families are shown in the figure.



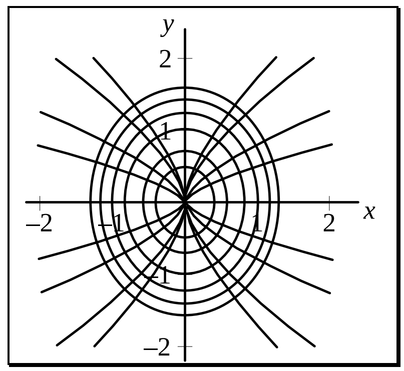
Orthogonal trajectories

54. $x^2 = 4cy^3$. Here

$$f(x, y) = \frac{x^2}{4y^3}$$

and $f_x = \frac{x}{2y^3}$, $f_y = -\frac{3x^2}{4y^4}$. The differential equation of the orthogonal family is

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{-3x}{2y}$$



Orthogonal trajectories

or $2y\,dy = -3x\,dx$, which has the general solution $2y^2 + 3x^2 = C$, where C is any real constant. These orthogonal families are shown in the figure.

55. $x^2 + y^2 = cy$. Here $f(x, y) = \frac{x^2 + y^2}{y}$, so

$$f_x = \frac{2x}{y},\ f_y = \frac{y^2 - x^2}{y^2}.$$

The differential equations of the orthogonal family are

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{\frac{y^2 - x^2}{y^2}}{\frac{2x}{y}} = \frac{y^2 - x^2}{2xy}.$$

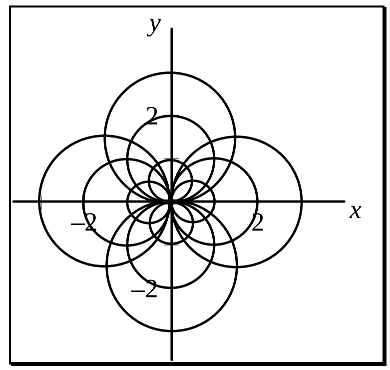
We are unable to solve this equation analytically, so we use a different approach inspired by looking at the graph of the original family, which consists of circles passing through the origin with centers on the y-axis.

Completing the square of the original equation, we can write $x^2 + y^2 = cy$ as $x^2 + \left(y - \frac{c}{2}\right)^2 = \frac{c^2}{4}$, which confirms the description and locates the centers at $\left(0,\ \frac{c}{2}\right)$.

We propose that the orthogonal family to the original family consists of another set of circles, $\left(x - \frac{C}{2}\right)^2 + y^2 = \frac{C^2}{4}$ centered at $\left(\frac{C}{2},\ 0\right)$ and passing through the origin.

To verify this conjecture we rewrite this equation for the second family of circles as $x^2 + y^2 = Cx$, which gives $C = g(x,\ y) = \frac{x^2 + y^2}{x}$ or $g_x = \frac{x^2 - y^2}{x^2}$, $g_y = \frac{2y}{x}$. Hence the proposed second family satisfies the equation

$$\frac{dy}{dx} = \frac{g_y}{g_x} = \frac{2xy}{x^2 - y^2},$$



Orthogonal circles

which indeed shows that the slopes are perpendicular to those of the original family derived above. Hence the original family of circles (centered on the *y*-axis) and the second family of circles (centered on the *x*-axis) are indeed orthogonal. These families are shown in the figure.

■ The Sine Function

56. The general equation is

$$y^2 + (y')^2 = 1$$

or

$$\frac{dy}{dx} = \pm\sqrt{1 - y^2}\ .$$

Separating variables and integrating, we get

$$\pm\sin^{-1} y = x + c \text{ or } y = \sin\left(\pm(x + c)\right) = \pm\sin(x + c).$$

This is the most general solution. Note that cos *x* is included because $\cos x = \sin\left(x - \frac{\pi}{2}\right)$.

■ Disappearing Mothball

57. (a) We have $\frac{dV}{dt} = -kA$, where V is the volume, t is time, A is the surface area, and k is a positive constant. Because $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$, the differential equation becomes

$$4\pi r^2 \frac{dr}{dt} = -4k\pi r^2$$

or

$$\frac{dr}{dt} = -k .$$

Integrating, we find $r(t) = -kt + c$. At $t = 0$, $r = \frac{1}{2}$; hence $c = \frac{1}{2}$. At $t = 6$, $r = \frac{1}{4}$; hence $k = \frac{1}{24}$, and the solution is

$$r(t) = -\frac{1}{24}t + \frac{1}{2},$$

where t is measured in months and r in inches. Because we can't have a negative radius or time, $0 \le t \le 12$.

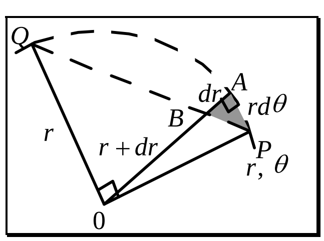
(b) Solving $-\frac{1}{24}t + \frac{1}{2} = 0$ gives $t = 12$ months or one year.

■ Four-Bug Problem

58. (a) According to the hint, the distance between the bugs is shrinking at the rate of 1 inch per second, and the hint provides an adequate explanation why this is so. Because the bugs are L inches apart, they will collide in L seconds. Because their motion is constantly towards each other and they start off in symmetric positions, they must collide at a point equidistant from all the bugs (i.e., the center of the carpet).

(b) The bugs travel at 1 inch per second for L seconds, hence the bugs travel L inches each.

(c)



This sketch of text Figure 1.3.8(b) shows a typical bug at $P = (r, \theta)$ and its subsequent position $A_x(r + dr, \theta + d\theta)$ as it heads toward the next bug at $Q = \left(r, \theta + \frac{\pi}{2}\right)$. Note that dr is negative, and consider that $d\theta$ is a very *small* angle, exaggerated in the drawing.

Consider the small shaded triangle *ABP*. For small $d\theta$:

- angle *BAP* is approximately a right angle,
- angle $APB = \text{angle } OQP = \dfrac{\pi}{4}$,
- side *BP* lies along *QP*.

Hence triangle *ABP* is similar to triangle *OQP*, which is a right isosceles triangle, so $-dr \approx r d\theta$.

Solving this separable DE gives $r = ce^{-\theta}$, and the initial condition $r(0) = 1$ gives $c = 1$. Hence our bug is following the path $r = e^{-\theta}$, and the other bugs' paths simply shift θ by $\dfrac{\pi}{e}$ for each successive bug.

■ Radiant Energy

59. Separating variables, we can write $\dfrac{dT}{T^4 - M^4} = -kdt$. We then write

$$\frac{1}{T^4 - M^4} = \frac{1}{(T^2 + M^2)(T^2 - M^2)} = \frac{1}{2M^2(T^2 - M^2)} - \frac{1}{2M^2(T^2 + M^2)}.$$

Integrating

$$\left\{\frac{1}{T^2 - M^2} - \frac{1}{T^2 + M^2}\right\}dT = -2kM^2 dt,$$

we find the implicit solution

$$\frac{1}{2M}\ln\left|\frac{M - T}{M + T}\right| - \frac{1}{M}\tan^{-1}\left(\frac{T}{M}\right) = -2kM^2 t + c$$

or in the more convenient form

$$\ln\left|\frac{M + T}{M - T}\right| + 2\arctan\left(\frac{T}{M}\right) = 4kM^3 t + C.$$

■ Suggested Journal Entry

60. Student Project

1.4 Euler's Method: Numerical Analysis [redacted]

■ **Easy by Calculator** $y' = \frac{t}{y}$, $y(0) = 1$

1. (a) Using step size 0.1 we enter t_0 and y_0, then calculate row by row to fill in the following table:

Euler's Method $(h = 0.1)$

n	$t_n = t_{n-1} + h$	$y_n = y_{n-1} + hy'_{n-1}$	$y'_n = \frac{t_n}{y_n}$
0	0	**1**	$\frac{0}{1} = 0$
1	0.1	1	$\frac{0.1}{1} = 0.1$
2	0.2	**1.01**	$\frac{0.2}{1.01} = 0.1980$
3	0.3	**1.0298**	$\frac{0.3}{1.0298} = 0.2913$

The requested approximations at $t = 0.2$ and $t = 0.3$ are $y_2(0.2) \approx 1.01$, $y_3(0.3) \approx 1.0298$.

(b) Using step size 0.05, we recalculate as in (a), but we now need twice as many steps. We get the following results.

Euler's Method $(h = 0.05)$

n	t_n	y_n	y'_n
0	**0**	**1**	0
1	0.05	1	0.05
2	0.1	1.0025	0.0998
3	0.15	1.0075	0.1489
4	**0.2**	**1.0149**	0.1971
5	0.25	1.0248	0.2440
6	**0.3**	**1.03698**	0.2893

The approximations at $t = 0.2$ and $t = 0.3$ are now $y_4(0.2) \approx 1.0149$, $y_6(0.3) \approx 1.037$.

(c) Solving the IVP $y' = \frac{t}{y}$, $y(0) = 1$ by separation of variables, we get $y\,dy = t\,dt$. Integration gives

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + c .$$

The initial condition $y(0) = 1$ gives $c = \frac{1}{2}$ and the implicit solution $y^2 - t^2 = 1$. Solving for y gives the explicit solution

$$y(t) = \sqrt{1+t^2} .$$

To four decimal place accuracy, the exact solutions are $y(0.2) = 1.0198$ and $y(0.3) =$ 1.0440. Hence, the errors in Euler approximation are

$$\begin{aligned} h = 0.1: \quad & \text{error} = y(0.2) - y_2(0.2) = 1.0198 - 1.0100 = 0.0098, \\ & \text{error} = y(0.3) - y_3(0.3) = 1.0440 - 1.0298 = 0.0142, \\ h = 0.05: \quad & \text{error} = y(0.2) - y_4(0.2) = 1.0198 - 1.0149 = 0.0050, \\ & \text{error} = y(0.3) - y_6(0.3) = 1.0440 - 1.0370 = 0.007 \end{aligned}$$

Euler approximations are both high, but the smaller stepsize gives smaller error.

■ **Calculator Again** $y' = ty$, $y(0) = 1$

2. (a) For each value of h we calculate a table as in Problem 1, with $y' = ty$. The results are summarized as follows.

Euler's Method

Comparison of Step Sizes

$\boldsymbol{h = 1}$		$\boldsymbol{h = 0.5}$		$\boldsymbol{h = 0.25}$		$\boldsymbol{h = 0.125}$	
t	***y* ≈**	***t***	***y* ≈**	***t***	***y* ≈**	***t***	***y* ≈**
0	1	0	1	0	1	0	1
1	**1**	0.5	1	0.25	1	0.125	1
		1	**1.25**	0.50	1.062	0.250	1.0156
				0.75	1.195	0.375	1.0474
				1	**1.419**	0.50	1.0965
						0.625	1.1650
						0.750	1.2560
						0.875	1.3737
						1	**1.5240**

(b) Solve the IVP $y' = ty$, $y(0)=1$ by separating variables to get $\frac{dy}{y} = tdt$. Integration yields $\ln|y| = \frac{t^2}{2} + c$, or $y = Ce^{t^2/2}$. Using the initial condition $y(0)=1$ gives the exact solution $y(t) = e^{t^2/2}$, so $y(1) = e^{1/2} \approx 1.6487$. Comparing with the Euler approximations gives

$$\begin{array}{lll} h=1: & \text{error} = 1.6487 - 1 & = 0.6487 \\ h=0.5: & \text{error} = 1.6487 - 1.25 & = 0.3987 \\ h=0.25: & \text{error} = 1.6487 - 1.419 & = 0.2297 \\ h=0.125: & \text{error} = 1.6487 - 1.524 & = 0.1247 \end{array}$$

■ **Computer Help Advisable**

3. $y' = 3t^2 - y$, $y(0)=1$; $[0, 1]$. Using a spreadsheet and Euler's method we obtain the following values:

Spreadsheet Instructions for Euler's Method

	A	B	C	D
1	n	t_n	$y_n = y_{n-1} + hy'_{n-1}$	$3t_n^2 - y_n$
2	0	0	1	$=3*t*B2\wedge 2-C$
3	$=A2+1$	$=B2+.1$	$=C2+.1*D2$	

Using step size $h = 0.1$ and Euler's method we obtain the following results.

Euler's Method $(h = 0.1)$

t	***y* ≈**	***t***	***y* ≈**
0	**1**	0.6	0.6822
0.1	0.9	0.7	0.7220
0.2	0.813	0.8	0.7968
0.3	0.7437	0.9	0.9091
0.4	0.6963	**1.0**	**1.0612**
0.5	0.6747		

Smaller steps give higher approximate values $y_n(t_n)$. The DE is not separable so we have no exact solution for comparison.

4. $y' = t^2 + e^{-y}$, $y(0) = 0$; $[0,\ 2]$

Using step size $h = 0.01$, and Euler's method we obtain the following results. (Table shows only selected values.)

Euler's Method $(\boldsymbol{h} = \mathbf{0.01})$			
t	***y* ≈**	***t***	***y* ≈**
0	**0**	1.2	1.2915
0.2	0.1855	1.4	1.6740
0.4	0.3568	1.6	2.1521
0.6	0.5355	1.8	2.7453
0.8	0.7395	**2.0**	**3.4736**
1.0	0.9858		

Smaller steps give higher approximate values $y_n(t_n)$. The DE is not separable so we have no exact solution for comparison.

5. $y' = \sqrt{t+y}$, $y(1) = 1$; $[1,\ 5]$

Using step size $h = 0.01$ and Euler's method we obtain the following results. (Table shows only selected values.)

Euler's Method $(\boldsymbol{h} = \mathbf{0.01})$			
t	***y***	***t***	***y***
1	**1**	3.5	6.8792
1.5	1.8078	4	8.5696
2	2.8099	4.5	10.4203
2.5	3.9942	**5**	**12.4283**
3	5.3525		

Smaller steps give higher $y_n(t_n)$. The DE is not separable so we have no exact solution for comparison.

6. $y' = t^2 - y^2$, $y(0) = 1$; $[0,\ 5]$

Using step size $h = 0.01$ and Euler's method we obtain following results. (Table shows only selected values.)

Euler's Method $(h = 0.01)$			
t	***y***	***t***	***y***
0	**1**	3	2.8143
0.5	0.6992	3.5	3.3464
1	0.7463	4	3.8682
1.5	1.1171	4.5	4.3843
2	1.6783	**5**	**4.8967**
2.5	2.2615		

Smaller steps give higher approximate values $y_n(t_n)$. The DE is not separable so we have no exact solution for comparison.

7. $y' = t - y$, $y(0) = 2$

Using step size $h = 0.05$ and Euler's method we obtain the following results. (Table shows only selected values.)

Euler's Method $(h = 0.05)$			
t	***y*** $\approx$	***t***	***y*** $\approx$
0	**2**	0.6	1.2211
0.1	1.8075	0.7	1.1630
0.2	1.6435	0.8	1.1204
0.3	1.5053	0.9	1.0916
0.4	1.3903	**1**	**1.0755**
0.5	1.2962		

Smaller steps give higher $y_n(t_n)$. The DE is not separable so we have no exact solution for comparison.

8. $y' = -\frac{t}{y}, \; y(0) = 1$

Using step size $h = 0.1$ and Euler's method we obtain the following results.

Euler's Method $(h = 0.1)$			
t	***y* ≈**	***t***	***y* ≈**
0	**1**	0.6	0.8405
0.1	1.0000	0.7	0.7691
0.2	0.9900	0.8	0.6781
0.3	0.9698	0.9	0.5601
0.4	0.9389	**1**	**0.3994**
0.5	0.8963		

The analytical solution of the initial-value problem is

$$y(t) = \sqrt{1 - t^2},$$

whose value at $t = 1$ is $y(1) = 0$. Hence, the absolute error at $t = 1$ is 0.3994. (Note, however, that the solution to this IVP does not exist for $t > 1$.) You can experiment yourself to see how this error is diminished by decreasing the step size or by using a more accurate method like the Runge-Kutta method.

9. $y' = \frac{\sin y}{t}, \; y(2) = 1$

Using step size $h = 0.05$ and Euler's method we obtain the following results. (Table shows only selected values.)

Euler's Method $(h = 0.05)$			
t	***y* ≈**	***t***	***y* ≈**
2	**1**	2.6	1.2366
2.1	1.0418	2.7	1.2727
2.2	1.0827	2.8	1.3079
2.3	1.1226	2.9	1.3421
2.4	1.1616	**3**	**1.3755**
2.5	1.1995		

Smaller stepsize predicts lower value.

10. $y' = -ty$, $y_0 = 1$

Using step size $h = 0.01$ and Euler's method we obtain the following results. (Table shows only selected values.)

Euler's Method $(h = 0.01)$			
t	***y* ≈**	***t***	***y* ≈**
0	**1**	0.6	0.8375
0.1	0.9955	0.7	0.7850
0.2	0.9812	0.8	0.7284
0.3	0.9574	0.9	0.6692
0.4	0.9249	**1**	**0.6086**
0.5	0.8845		

Smaller step size predicts lower value. The analytical solution of the initial-value problem is

$$y(t) = e^{-t^2/2}$$

whose exact value at $t = 1$ is $y(1) = 0.6065$. Hence, the absolute error at $t = 1$ is

$$\text{error} = |0.6065 - 0.6086| = 0.0021.$$

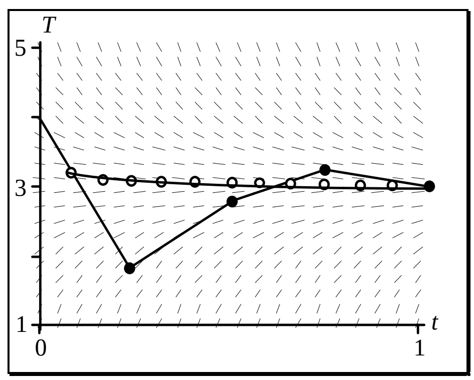
■ **Stefan's Law Again** $\frac{dT}{dt} = 0.05\left(3^4 - T^4\right),\ T(0) = 4$.

11. (a)

Euler's Method

$\boldsymbol{h = 0.25}$			$\boldsymbol{h = 0.1}$		
n	t_n	T_n	n	t_n	T_n
0	**0.00**	**4.0000**	**0**	**0.00**	**4.0000**
1	0.25	1.8125	1	0.10	3.1250
2	0.50	2.6901	2	0.20	3.0532
3	0.75	3.0480	3	0.30	3.0237
4	**1.00**	**2.9810**	4	0.40	3.0107
			5	0.50	3.0049
			6	0.60	3.0023
			7	0.70	3.0010
			8	0.80	3.0005
			9	0.90	3.0002
			10	**1.00**	**3.0001**

(b) The graph shows that the larger step approximation (black dots) overshoots the mark but recovers, while the smaller step approximation (white dots) avoids that problem.



(c) There is an equilibrium solution at $T = 3$, which is confirmed both by the direction field and the slope $\frac{dT}{dt}$. This is an exact solution that both Euler approximations get very close to by the time $t = 1$.

■ **Nasty Surprise**

12. $y' = y^2$, $y(0) = 1$

Using Euler's method with $h = 0.25$ we obtain the following values.

Euler's Method $(h = 0.25)$		
t	$y \approx$	$y' = y^2$
0	**1**	1
0.25	1.25	1.5625
0.50	1.6406	2.6917
0.75	2.3135	5.3525
1.00	**3.6517**	

Euler's method estimates the solution at $t = 1$ to be 3.6517, whereas from the analytical solution $y(t) = \frac{1}{1-t}$, or from the direction field, we can see that the solution blows up at 1. So Euler's method gives an approximation far too small.

■ **Approximating *e***

13. $y' = y$, $y(0) = 1$

Using Euler's method with different step sizes h, we have estimated the solution of this IVP at $t = 1$. The true value of $y = e^t$ for $t = 1$ is $e \approx 2.7182818\ldots$.

Euler's Method		
h	$y(1) \approx$	$e - y(1)$
0.5	2.25	0.4683
0.1	2.5937	0.1245
0.05	2.6533	0.0650
0.025	2.6850	0.0332
0.01	2.7048	0.0135
0.005	2.7115	0.0068
0.0025	2.7149	0.0034
0.001	2.7169	0.0013

We now use the fourth-order Runge-Kutta method with the same values of *h*, getting the following values.

Runge-Kutta Method

h	$y(1)$	$e - y(1)$
0.5	2.717346191	0.00093
0.1	2.718279744	0.21×10^{-5}
0.05	2.718281693	0.13×10^{-6}
0.025	2.718281820	0.87×10^{-8}
0.01	2.718281828	0.22×10^{-11}

Note that even with a large step size of $h = 0.5$ the Runge-Kutta method gives $y(1)$ correct to within 0.001, which is better than Euler's method with stepsize $h = 0.001$.

■ Double Trouble or Worse

14. $y = y^{1/3}$, $y(0) = 0$

(a) The solution starting at the initial point $y(0) = 0$ never gets off the ground (i.e., it returns all zero values for y_n). For this IVP, $y_n(6) = 0$.

(b) Starting with $y(0) = 0.01$, the solution increases. We have given a few values in the following table and see that $y_n(6) \approx 7.9134$.

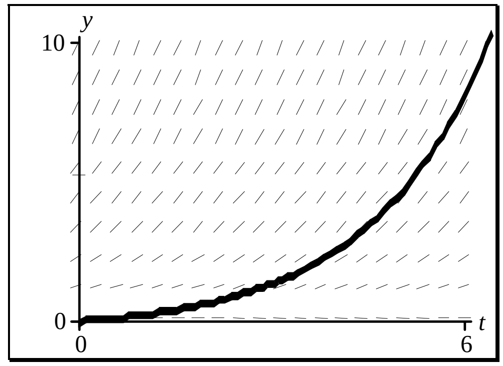
Euler's Method $y' = y^{1/3}$, $y(0) = 0.01$ ($h = 0.1$)

t	y	t	y
0	0.01	3.5	3.5187
0.5	0.2029	4	4.3005
1	0.5454	4.5	5.1336
1.5	0.9913	5	6.0151
2	1.5213	5.5	6.9424
2.5	2.1241	**6**	**7.9134**
3	2.7918		

(c) The direction field of $y' = y^{1/3}$ for

$$0 \le t \le 6, \; 0 \le y \le 10$$

confirms the values found in (b).



$y' = \sqrt[3]{y}$

■ **Roundoff Problems**

15. If a roundoff error of ε occurs in the initial condition, then the solution of the new IVP $y' = y$, $y(0) = A + \varepsilon$ is

$$y(t) = (A + \varepsilon)e^t = Ae^t + \varepsilon e^t.$$

The difference between this perturbed solution and Ae^t is εe^t. This difference at various intervals of time will be

$$t = 1 \Rightarrow \text{difference} = \varepsilon e$$
$$t = 10 \Rightarrow \text{difference} = \varepsilon e^{10} \approx 22{,}026\varepsilon$$
$$t = 20 \Rightarrow \text{difference} = \varepsilon e^{20} = 485{,}165{,}195\varepsilon.$$

Hence, the accumulate roundoff error grows at an exponential rate.

■ **Think Before You Compute**

16. Because $y = 2$ and $y = -2$ are constant solutions, any initial conditions starting at these values should remain there. On the other hand, a roundoff error in computations starting near $y = -2$ is not as serious as near $y = 2$, because near $y = -2$ the perturbed solution will move *towards* the *stable* solution –2.

■ Runge-Kutta Method

17. $y' = t + y$, $y(0) = 0$, $h = 1$

(a) By Euler's method,

$y_1 = y_0 + h(t_0 + y_0) = 0$

By 2nd order Runge Kutta

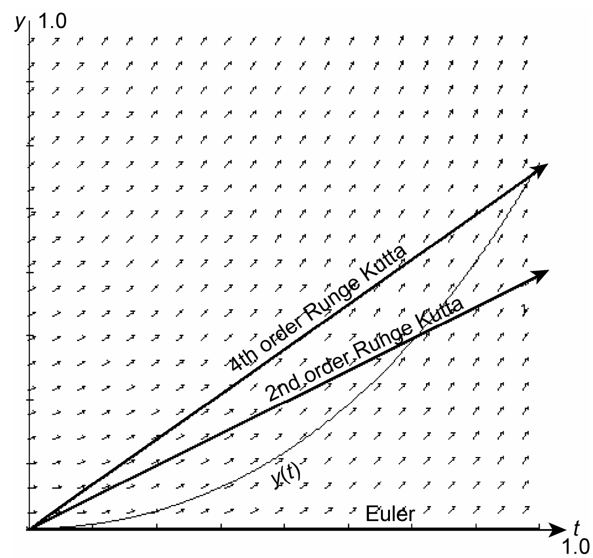
$y_1 = y_0 + hk_{02}$,

$k_{01} = t_0 + y_0 = 0$

$$k_{02} = \left(t_0 + \frac{h}{2}\right) + \left(y_0 + \frac{h}{2}k_{01}\right)$$

$$= \frac{1}{2} + 0$$

$$y_1 = 0 + \left[\frac{1}{2}\right] = \frac{1}{2} = 0.5$$



By 4th order Runge Kutta.

$$y_1 = y_0 + \frac{h}{6}\left(k_{01} + 2k_{02} + 2k_{03} + k_{04}\right)$$

$k_{01} = t_0 + y_0 = 0$

$$k_{02} = \left(t_0 + \frac{h}{2}\right) + \left(y_0 + \frac{h}{2}k_{01}\right) = \frac{1}{2}$$

$$k_{03} = \left(t_0 + \frac{h}{2}\right) + \left(y_0 + \frac{h}{2}k_{02}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$k_{04} = (t_0 + h) + \left(y_0 + \frac{h}{2}k_{03}\right) = 1 + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = 1.375$$

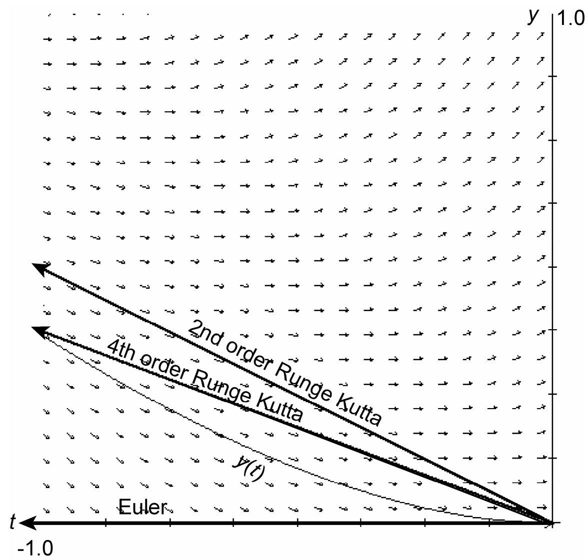
$$y_1 = 0 + \frac{1}{6}\left(0 + 2\left(\frac{1}{2}\right) + 2\left(\frac{3}{4}\right) + 1.375\right) = \frac{1}{6}(3.875) \approx 0.646$$

(b) Second-order Runge Kutta is much better than Euler for a single step approximation, but fourth-order RK is almost right on (*slightly* low).

(c) If $y(t) = -t - 1 + e^t$,

then $y(1) = -2 + e \approx 0.718$.

18. $y' = t + y,\ y(0) = 0,\ h = -1$



(a) By Euler's method,

$y_1 = y_0 + h(t_0 + y_0) = 0$

By 2nd order Runge Kutta

$y_1 = y_0 + hk_{02},$

$k_{01} = t_0 + y_0 = 0$

$k_{02} = \left(t_0 + \frac{h}{2}\right) + \left(y_0 + \frac{h}{2}k_{01}\right) = -\frac{1}{2}$

$y_1 = y_0 - 1\left(-\frac{1}{2}\right) = 0.5$

By 4th order Runge Kutta.

$y_1 = y_0 + \frac{h}{6}\left(k_{01} + 2k_{02} + 2k_{03} + k_{04}\right)$

$k_{01} = t_0 + y_0 = 0$

$k_{02} = \left(t_0 + \frac{h}{2}\right) + \left(y_0 + \frac{h}{2}k_{01}\right) = -\frac{1}{2} = -0.5$

$k_{03} = \left(t_0 + \frac{h}{2}\right) + \left(y_0 + \frac{h}{2}k_{02}\right) = -\frac{1}{2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{1}{4} = -0.25$

$k_{04} = (t_0 + h) + \left(y_0 + \frac{h}{2}k_{03}\right) = -1 + \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right) = -\frac{7}{8} = -0.875$

$y_1 = 0 + -\frac{1}{6}\left(0 + 2\left(-\frac{1}{2}\right) + 2\left(-\frac{1}{4}\right) + -\frac{7}{8}\right) = -\frac{1}{6}(-2.375) \approx 0.396$

(b) Second-order Runge Kutta is *high* though closer than Euler. Fourth order R-K is *very* close.

(c) If $y(t) = -t - 1 + e^t$,

then $y(-1) = e^{-1} \approx 0.368.$

■ **Runge-Kutta vs. Euler**

19. $y' = 3t^2 - y$, $y(0) = 1$; [0, 1]

Using the fourth-order Runge-Kutta method and $h = 0.1$ we arrive at the following table of values.

Runge-Kutta Method, $y' = 3t^2 - y$, $y(0) = 1$

t	*y*	*t*	*y*
0	1	0.6	0.7359
0.1	0.9058	0.7	0.7870
0.2	0.8263	0.8	0.8734
0.3	0.7659	0.9	0.9972
0.4	0.7284	1.0	1.1606
0.5	0.7173		

We compare this with #3 where Euler's method gave $y(1) \approx 1.0612$ for $h = 0.1$. Exact solution by separation of variables is not possible.

20. $y' = t - y$, $y(0) = 2$

Using the fourth-order Runge-Kutta method and $h = 0.1$ we arrive at the following table of values.

Runge-Kutta Method, $y' = t - y$, $y(0) = 2$

t	*y*	*t*	*y*
0	2	0.6	1.2464
0.1	1.8145	0.7	1.1898
0.2	1.6562	0.8	1.148
0.3	1.5225	0.9	1.1197
0.4	1.4110	1.0	1.1036
0.5	1.3196		

We compare this with #7 where Euler's method gives $y(1) \approx 1.046$ for step $h = 0.1$; $y(1) \approx 1.07545$ for step $h = 0.05$. Exact solution by separation of variables is not possible.

21. $y' = -\frac{t}{y}$, $y(0) = 1$

Using the fourth-order Runge-Kutta method and $h = 0.1$ we arrive at the following table of values.

Runge-Kutta Method, $y' = -\frac{t}{y}$, $y(0) = 1$

t	*Y*	*t*	*y*
0	1	0.6	0.8000
0.1	0.9950	0.7	0.7141
0.2	0.9798	0.8	0.6000
0.3	0.9539	0.9	0.4358
0.4	0.9165	1.0	0.04880
0.5	0.8660		

We compare this with #8 where Euler's method for step $h = 0.1$ gave $y(1) \approx 0.3994$, and the exact solution $y(t) = \sqrt{1-t^2}$ gave $y(1) = 0$. The Runge-Kutta approximate solution is much closer to the exact solution.

22. $y' = -ty$, $y(0) = 1$

Using the 4th-order Runge Kutta method and $h = 0.01$ to arrive at the following table. (Table shows only selected values.)

Runge-Kutta Method, $y' = -ty$, $y(0) = 1$

t	*y*	*t*	*y*
0	**1**	0.6	0.8353
0.1	0.9950	0.7	0.7827
0.2	0.9802	0.8	0.7261
0.3	0.9560	0.9	0.6670
0.4	0.9231	1	0.6065
0.5	0.8825		

We compare this with #10 where Euler's method for step $h = 0.1$ gave $y(1) \approx 0.6086$, and the exact solution $y(t) = e^{-t^2/2}$ gave $y(1) = 0.6065$. The Runge-Kutta approximate solution is exact within given accuracy.

■ Euler's Errors

23. (a) Differentiating $y' = f(t, y)$ gives

$$y'' = f_t + f_y y' = f_t + f_y f \ .$$

Here we assume f_t, f_y and $y' = f$ are continuous, so y'' is continuous as well.

(b) The expression

$$y(t_n + h) = y(t_n) + y'(t_n)h + \frac{1}{2}y''(t_n^*)h^2$$

is simply a statement of Taylor series to first degree, with remainder.

(c) Direct computation gives

$$e_{n+1} \le M\frac{h^2}{2} \, .$$

(d) We can make the local discretization error e_n in Taylor's method less than a preassigned value E by choosing h so it satisfies $e_n \le \frac{Mh^2}{2} \le E$, where M is the maximum of the second derivative of y'' on the interval $[t_n, t_{n+1}]$. Hence, if $h \le \frac{\sqrt{2E}}{M}$, we have the desired condition $e_n \le E$.

■ Three-Term Taylor Series

24. (a) Starting with $y' = f(t, y)$, and differentiating with respect to t, we get

$$y'' = f_t(t, y) + f_y(t, y)y' = f_t(t, y) + f_y(t, y)f(t, y) \, .$$

Hence, we have the new rule

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{1}{2}h^2\left[f_t(t_n, y_n) + f_y(t_n, y_n)f(t_n, y_n)\right].$$

(b) The local discretization error has order of the highest power of h in the remainder for the approximation of y_{n+1}, which in this case is 3.

(c) For the equation $y' = f(t, y) = \frac{t}{y}$ we have $f_t(t, y) = \frac{1}{y}$, $f_y(t, y) = -\frac{t}{y^2}$ and so the preceding three-term Taylor series becomes

$$y_{n+1} = y_n + h\left(\frac{t_n}{y_n}\right) + \frac{1}{2}h^2\left[\frac{1}{y_n} - \frac{t_n^2}{y_n^3}\right].$$

Using this formula and a spreadsheet we get the following results.

Taylor's Three-Term Series

Approximation of $y' = \frac{t}{y}$, $y(0) = 1$

t	***y***	***t***	***y***
0	**1**	0.6	1.1667
0.1	1.005	0.7	1.2213
0.2	1.0199	0.8	1.2314
0.3	1.0442	0.9	1.3262
0.4	1.0443	**1.0**	**1.4151**
0.5	1.1185		

The exact solution of the initial-value problem $y' = \frac{t}{y}$, $y(0) = 1$ is $y(t) = \sqrt{1+t^2}$, so we have $y(1) = \sqrt{2} \approx 1.4142\ldots$. Taylor's three-term method gave the value 1.4151, which has an error of

$$\left|\sqrt{2} - 1.4151\right| \approx 0.0009 .$$

(d) For the differential equation $y' = f(t,\ y) = ty$ we have $f_t(t,\ y) = y$, $f_y(t,\ y) = t$, so the Euler three-term approximation becomes

$$y_{n+1} = y_n + ht_n y_n + \frac{1}{2}h^2\left[y_n - t_n^2 y_n\right].$$

Using this formula and a spreadsheet, we arrive at the following results.

Taylor's Three-Term Series

Approximation of $y' = ty$, $y(0) = 1$

t	***Y***	***t***	***y***
0	**1**	0.6	1.1962
0.1	1.005	0.7	1.2761
0.2	1.0201	0.8	1.3749
0.3	1.0458	0.9	1.4962
0.4	1.1083	**1.0**	**1.6444**
0.5	1.1325		

The solution of $y' = ty$, $y(0) = 1$ is $y(t) = e^{t^2/2}$, so $y(1) = \sqrt{e} \approx 1.649\ldots$. Hence the error at $t = 1$ using Taylor's three-term method is

$$\left|\sqrt{e} - 1.6444\right| \approx 0.0043 .$$

■ Richardson's Extrapolation

Sharp eyes may have detected the elimination of absolute value signs when equation (7) is rewritten as equation (9). This is legitimate with no further argument if y' is positive and monotone increasing, as is the case in the suggested exercises.

25. $y' = y$, $y(0) = 1$.

Our calculations are listed in the following table. Note that we use $y_R(0.1)$ as initial condition for computing $y_R(0.2)$.

	One-step Euler	**Two-step Euler**	**Richardson approx.** $y_R(t^*) =$	**Exact solution**
t^*	$y(t^*, h)$	$y(t^*, h)$	$2y(t^*, h) - y(t^*, h)$	$y = e^t$
0.1	1.1	1.1025	**1.1050**	$e^{0.1} = 1.1052$
0.2	1.2155	1.2183	**1.2211**	$e^{0.2} = 1.2214$

26. $y' = ty$, $y(0) = 1$.

Our calculations are listed in the following table. Note that we use $y_R(0.1)$ as initial condition for computing $y_R(0.2)$.

	One-step Euler	**Two-step Euler**	**Richardson approx.** $y_R(t^*) =$	**Exact solution**
t^*	$y(t^*, h)$	$y(t^*, h)$	$2y(t^*, h) - y(t^*, h)$	$y = e^{t^2}$
0.1	1.0	1.0025	**1.005**	$e^{0.01} = 1.0101$
0.2	1.01505	1.0176	**1.02005**	$e^{0.04} = 1.0408$

27. $y' = y^2$, $y(0) = 1$.

Our calculations are listed in the following table (on the next page). Note that we use $y_R(0.1)$ as initial condition for computing $y_R(0.2)$.

t^*	One-step Euler $y(t^*, h)$	Two-step Euler $y(t^*, h)$	Richardson approx. $y_R(t^*) = 2y(t^*, h) - y(t^*, h)$	Exact solution $y = 1/(1-t)$
0.1	1.1	1.1051	**1.1102**	1.1111
0.2	1.2335	1.2405	**1.2476**	1.2500

28. $y' = \sin(ty),\ y(0) = 1$.

Our calculations are listed in the following table. Note that we use $y_R(0.1)$ as initial condition for computing $y_R(0.2)$.

t^*	One-step Euler $y(t^*, h)$	Two-step Euler $y(t^*, h)$	Richardson approx. $y_R(t^*) = 2y(t^*, h) - y(t^*, h)$	Exact solution no formula
0.1	1.1	1.0025	**1.0050**	
0.2	1.0150	1.0176	**1.0201**	1.02013 by Runge-Kutta

■ **Integral Equation**

29. (a) Starting with

$$y(t) = y_0 + \int_{t_0}^{t} f(s,\ y(s))\,ds$$

we differentiate respect to *t*, getting $y' = f(t,\ y(t))$. We also have $y(t_0) = y_0$.

Conversely, starting with the initial-value problem

$$y' = f(t,\ y(t)),\ y(t_0) = y_0$$

we integrate getting the solution

$$y(t) = \int_{t_0}^{t} f(s,\ y(s))\,ds + c\,.$$

Using the initial condition $y(t_0) = y_0$, gives the constant $c = y_0$. Hence, the integral equation is equivalent to IVP.

(b) The initial-value problem, $y' = f(t)$, $y(0) = y_0$, is transformed into the integral equation

$$y(t) = y_0 + \int_0^t f(s)\,ds .$$

To find the approximate value of the solution at $t = T$, we evaluate the preceding integral at $t = T$ using the Riemann sum with left endpoints, getting

$$\begin{aligned} y(T) &= y_0 + \int_0^T f(s)\,ds \\ &\approx y_0 + h\left[f(0) + f(h) + \ldots + f(T-h)\right]. \end{aligned}$$

If we, however, write the expression as

$$\begin{aligned} y(T) &= y_0 + h\left[f(0) + f(h) + \ldots + f(T-h)\right] \\ &= y_1 + hf(h) + \ldots + hf(T-h) \\ &= y_2 + hf(2h) + \ldots + hf(T-h) \\ &= y_3 + hf(3h) + \ldots + hf(T-h) + y_{n-1} + hf(T-h) \\ &\quad \ldots \qquad \ldots \\ &= y_{n-1} + h(T-h) \\ &= y_n . \end{aligned}$$

we get the desired conclusion.

(c) The Riemann sum only holds for integrals of the form $\int_a^b f(t)\,dt$.

■ Computer Lab: Other Methods

30. **Sample study of different numerical methods.** We solve the IVP of Problem 5 $y' = \sqrt{t+y}$, $y(1) = 1$ by several different methods using step size $h = 0.1$. The table shows a printout for selected values of y using one non-Euler method.

Fourth Order Runge-Kutta Method

t	*Y*	*t*	*y*
1	1	3.5	6.8910
1.5	1.8100	4	8.5840
2	2.8144	4.5	10.4373
2.5	4.0010	5	12.4480
3	5.3618		

We can now compare the following approximations for Problem 5:

Euler's method (answer in text)	$h = 0.1$	$y(5) \approx 12.2519$
Euler's method (solution in manual)	$h = 0.01$	$y(5) \approx 12.4283$
Runge-Kutta method (above)	$h = 0.1$	$y(5) \approx 12.4480$

We have no exact solution for Problem 5, but you might use step $h = 0.1$ to approximate $y(5)$ by other methods (for example Adams-Bashforth method or Dormand-Prince method) then explain which method seems most accurate. A graph of the direction field could give insight.

■ **Suggested Journal Entry I**

31. Student Project

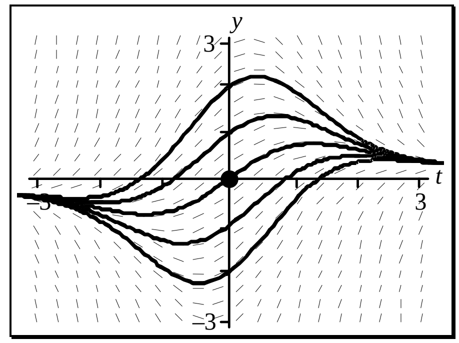
■ **Suggested Journal Entry II**

32. Student Project

1.5 Picard's Theorem: Theoretical Analysis

■ Picard's Conditions

1. (a) $y' = f(t,\ y) = 1 - ty$, $y(0) = 0$



Hence $f_y = -t$. The fact that f is continuous for all t tells us a solution exists passing through each point in the *ty* plane. The further fact that the derivative f_y is also continuous for all t and y tells us that the solution is unique. Hence, there is a unique solution of this equation passing through $y(0) = 0$. The direction field is shown in the figure.

(b) Picard's conditions hold in entire *ty* plane.

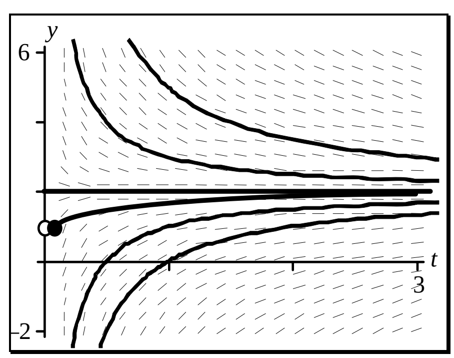
(c) Not applicable - the answer to part (a) is positive.

2. (a) $y' = \dfrac{2 - y}{t}$, $y(0) = 1$

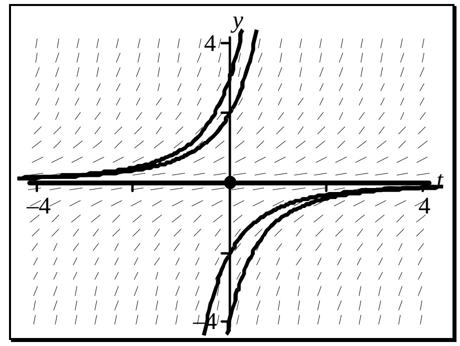
Here $f(t,\ y) = \dfrac{2 - y}{t}$, $f_y = -\dfrac{1}{t}$. The functions f and f_y are continuous for $t \neq 0$, so there is a unique solution passing through any initial point $y(t_0) = y_0$ with $t_0 \neq 0$. When $t_0 = 0$ the derivative y' is not only discontinuous, it isn't defined. No solution of this DE passes through points $(t_0,\ y_0)$ with $t_0 = 0$. In particular the DE with IC $y(0) = 1$ does not make sense.

(b) Uniqueness/existence in *either* the right half plane $t > 0$ *or* the left half plane $t < 0$; any rectangle that does not include $t = 0$ will satisfy Picard's Theorem.

(c) If we think of DEs as models for physical phenomena, we might be tempted to replace t_0 in the IC by a small number and examine the unique solution, which we know exists. It would also be useful to draw the direction field of this equation and see the big picture. The direction field is shown in the figure.



3. (a) $y' = y^{4/3},\ y(0) = 0$



Here

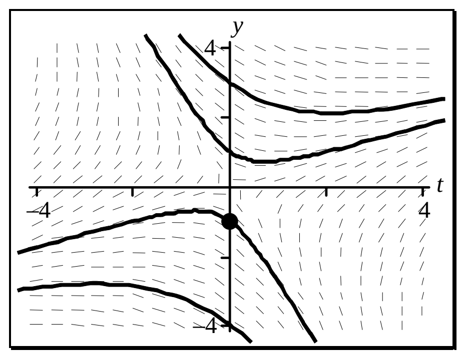
$$f(t,\ y) = y^{4/3}$$
$$f_y = \frac{4}{3} y^{1/3}.$$

Here f and f_y are continuous for all t and y, so by Picard's theorem we conclude that the DE has a unique solution through any initial condition $y(t_0) = y_0$. In particular, there will be a unique solution passing through $y(0) = 0$, which we know to be $y(t) \equiv 0$. The directions field of the equation is shown in the figure.

(b) Picard's conditions hold in entire *ty* plane.

(c) Not applicable - the answer to part (a) is positive.

4. (a) $y' = \dfrac{t-y}{t+y},\ y(0) = -1$



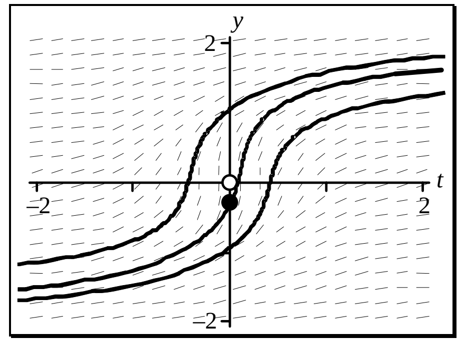
Here both

$$f(t,\ y) = \frac{t-y}{t+y}$$
$$f_y = -\frac{2t}{(t+y)^2}$$

are continuous for t and y except when $y = -t$. Hence, there is a unique solution passing through any initial condition $y(t_0) = y_0$ as long as $y_0 \neq -t_0$. When $y = -t$ the derivative y' is not only discontinuous but also not even defined, so there is really no need to resort to Picard's theorem to conclude there is no solution passing through such points.

(b), (c) Picard's conditions hold for the entire *ty* plane except the line $y = -t$, so any rectangle that does not include any part of $y = -t$ satisfies Picard's Theorem.

5. (a) $y' = \dfrac{1}{t^2 + y^2}$, $y(0) = 0$



Here both

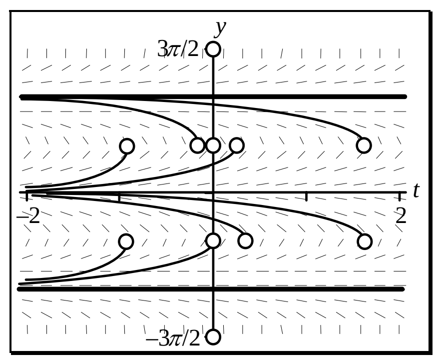
$$f(t,\ y) = \frac{1}{t^2 + y^2}$$
$$f_y(t,\ y) = -\frac{2y}{\left(t^2 + y^2\right)^2}$$

are continuous for all t and y except at the point $y = t = 0$. Hence, there is a unique solution passing through any initial point $y(t_0) = y_0$ except $y(0) = 0$. In this case f does not exist, so the IVP does not make sense. The direction field of the equation illustrates these ideas (see figure).

(b) Picard's Theorem gives existence/uniqueness for any rectangle that does not include the origin.

(c) It may be useful to replace the initial condition $y(0) = 0$ by $y(0) = y_0$ with small but nonzero y_0.

6. (a) $y' = \tan y$, $y(0) = \dfrac{\pi}{2}$



Here

$$f(t,\ y) = \tan y$$
$$f_y = \sec^2 y$$

are both continuous except at the points

$$y = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2}, \dots\ .$$

Hence, there exists a unique solution passing through $y(t_0) = y_0$ except when

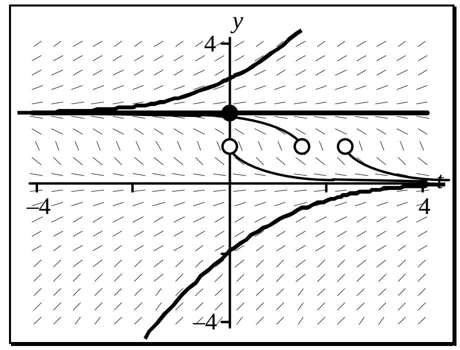
$$y = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \dots\ .$$

The IVP problem passing through $\dfrac{\pi}{2}$ does not have a solution. It would be useful to look at the direction field to get an idea of the behavior of solutions for nearby initial points. The direction field of the equation shows that where Picard's Theorem does not work the slope has become vertical (see figure).

(b) Existence/uniqueness conditions are satisfied over any rectangle with y-values *between* two successive odd multiples of $\frac{\pi}{2}$.

(c) There are no solutions going forward in time from any points near $\left(0,\ \frac{\pi}{2}\right)$.

7. (a) $y' = \ln|y-1|,\ y(0) = 2$



Here

$$f(t,\ y) = \ln|y-1|$$

$$f_y = \frac{1}{|y-1|}$$

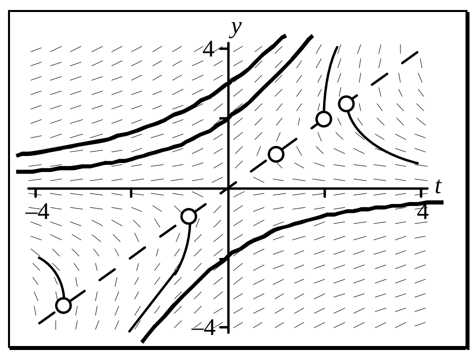
are both continuous for all t and y as long as

$$y \neq 1,$$

where neither is defined. Hence, there is a unique solution passing through any initial point $y(t_0) = y_0$ with $y_0 \neq 1$. In particular, there is a unique solution passing through $y(0) = 2$. The direction field of the equation illustrates these ideas (see figure).

(b), (c) The Picard Theorem holds for entire *ty* plane except the line $y = 1$.

8. (a) $y' = \frac{y}{y-t},\ y(1) = 1$



Here

$$f(t,\ y) = \frac{y}{y-t}$$

$$f_y = -\frac{t}{(y-t)^2}$$

are continuous for all t and y except when $y \neq t$ where neither function exists. Hence, we can be assured there is a unique solution passing through $y(t_0) = y_0$ except when $t_0 = y_0$. When $t_0 = y_0$ the derivative isn't defined, so IVP problems with these IC does not make sense. Hence the IVP with $y(1) = 1$ is not defined. See figure for the direction field of the equation.

(b) The Picard Theorem holds for the entire *ty* plane except the line $y = t$, so it holds for any rectangle that does not include any part of $y = t$.

(c) It may be useful to replace the initial condition $y(1)=1$ by $y(1)=1+\varepsilon$. However, you should note that the direction field shows that $\varepsilon>0$ will send solution toward ∞, $\varepsilon<0$ will send solution toward zero.

■ Linear Equations

9. $y'+p(t)y=q(t)$

For the first-order linear equation, we can write $y'=q(t)-p(t)y$ and so

$$f(t,\ y)=q(t)-p(t)y$$
$$f_y(t,\ y)=-p(t).$$

Hence, if we assume $p(t)$ and $q(t)$ are continuous, then Picard's theorem holds at any point $y(t_0)=y_0$.

■ Eyeballing the Flows

For the following problems it appears from the figures given in the text that:

10. A unique solution will pass through each point *A*, *B*, *C*, and *D* and the solutions appear to exist for all *t*.

11. A unique solution passes through *A* and *B* defined for negative *t*; no unique solution passes through *C* where the derivative is not uniquely defined; a unique solution passes through *D* for positive *t*.

12. Unique solutions exist passing through points *B* and *C* on intervals until the solution curve reaches the *t*-axis, where finite slope does not exist. Nonunique solutions at *A*; possibly unique solutions at *D* where $t=y=0$.

13. A unique solution will pass through each of the points *A*, *B*, *C*, and *D*. Solutions appear to exist for all *t*.

14. A unique solution will pass through each of the points *A*, *B*, *C*, and *D*. Solutions appear to exist for all *t*.

15. A unique solution will pass through each of the points *B*, *C*, and *D*. Solutions exist only for $t>t_A$ or $t<t_A$ because all solutions appear to leave from or go toward A, where there is no unique slope.

16. Unique solutions will pass through each of the points *A*, *B*, *C*, and *D*. Solutions appear to exist for all *t*.

17. A unique solution will pass through each of the points *A*, *B*, *C*, and *D*. Solutions appear to exist for all *t*.

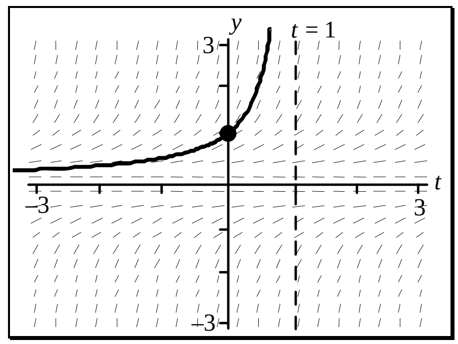
18. A unique solution will pass through each of the points *A*, *B*, *C*, and *D*. Solutions appear to exist for all *t*.

■ Local Conclusions

19. (a) $f(t,\ y) = y^2$, $f_y = 2y$, $y(0) = 1$

are both continuous for all *t*, *y* so by Picard's theorem there is a unique solution passing through any point *t*, *y*. Hence the existence and uniqueness conditions hold for any initial condition in the entire *ty* plane. However, this example exhibits an important weakness of Picard's Theorem: For any particular initial condition, the solution may not exist over the entire plane. In the given IVP the solution exists only for $t < 1$.

(b)



Solution of $y' = y^2$, $y(0) = 1$

(c) The separated equation is $y^{-2}dy = dt$. Integrating gives the result $-y^{-1} = t + c$ and solving for *y*, we get $-\dfrac{1}{t+c}$. Substituting the initial condition $y(0) = 1$, gives $c = -1$. Hence, we have $y(t) = \dfrac{1}{1-t}$, $t < 1$, $y > 0$. The interval over which this solution is defined cannot pass through $t = 1$, and the solution with IC $y(0) = 1$ exists on the interval $(-\infty,\ 1)$.

(d) Because Picard's theorem holds for all *t*, *y* we conclude there exists a unique solution to $y' = y^2$, $y(t_0) = y_0$ for any $(t_0,\ y_0)$. To find the size of the interval of existence, we must solve the IVP, getting

$$y(t) = -\frac{1}{t - t_0 - \frac{1}{y_0}}.$$

Hence, the interval over which this solution is defined cannot pass through $t = t_0 + \dfrac{1}{y_0}$, which implies an interval of

$$\left(-\infty,\ t_0 + \frac{1}{y_0}\right)$$

for positive y_0 and

$$\left(t_0 - \frac{1}{|y_0|},\ \infty\right)$$

for negative y_0.

■ **Nonuniqueness**

20. $y' = y^{1/3}$, $y(0) = 0$

Because $f = y^{1/3}$ is continuous for all $(t,\ y)$, Picard's theorem says that there exists a solution through any point $y(t_0) = y_0$. However, $f_y = \frac{1}{3}y^{-2/3}$ is not continuous when $y = 0$ so Picard's theorem does not guarantee a unique solution through any point where $y = 0$.

In fact we can find an infinite number of solutions passing through the origin. We first separate variables, getting $y^{-1/3}dy = dt$, and integrating gives

$$\frac{3}{2}y^{2/3} = t + c\ .$$

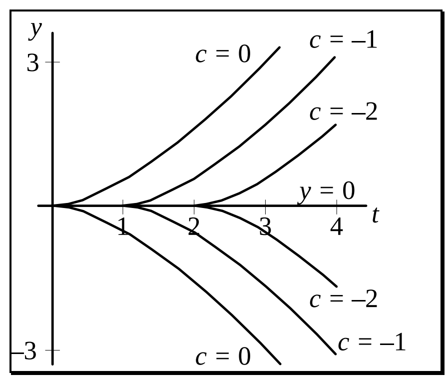
Picking the initial condition $y(0) = 0$, we find $c = 0$. Hence, we have found *one* solution of the initial-value problem as

$$y(t) = \pm\left(\frac{2}{3}\right)^{3/2} t^{3/2}\ .$$

But clearly, $y(t) \equiv 0$ is another solution. In fact, we can *paste* these solutions together at $t = 0$. Futhermore, we can also paste together $y = 0$ with infinitely many additional solutions, using any $c < 0$, getting an infinite number of solutions to the initial-value problem as

$$y(t) = \begin{cases} 0 & t < |c| \\ \pm\left(\frac{2}{3}\right)^{3/2}(t+c)^{3/2} & t \ge |c| \end{cases}$$

for any $c \le 0$. A few of these solutions are plotted (see figure).



Nonuniqueness of solutions through $y(0) = 0$

■ More Nonuniqueness

21. $y' = \sqrt{y}$, $y(0) = 0$, $t_0 > 0$

For $t < t_0$, the solution is $y(t) \equiv 0$. For $t > t_0$, we have $y = \frac{1}{4}(t - t_0)^2$.

At $t = t_0$ the left-hand derivative of $y(t) \equiv 0$ is 0, and the right-hand derivative of $y(t) = \frac{1}{4}(t - t_0)^2$ is 0, so they agree.

■ Seeing vs. Believing

22. No, the solution does not "merge" with $y = -1$.

Consider $y' = 3t^2(1 + y) = f(t, y)$. Note that $y = -1$ is an equilibrium solution.

We observe:

1. $f(t, y)$ is continuous for all t and y.
2. $\dfrac{\partial f}{\partial y} = 3t^2$ is continuous for all t and y

By Picard's Theorem, we know there is a unique solution through any initial point. Because the line $y = -1$ passes through every point with y-coordinate = 1, no other solution can merge with $y = -1$ and can only approach $y = -1$ asymptotically.

■ Converse of Picard's Theorem Fails

23. (a) Note that $\dfrac{dy}{dt} = |y| = f(t, y)$, so that $f(t, y) = \begin{cases} -y & y < 0 \\ y & y \geq 0 \end{cases}$

has a partial derivative $\dfrac{\partial f}{\partial y} = \begin{cases} -1 & y < 0 \\ 1 & y > 0, \end{cases}$

that is not continuous at $y = 0$. Consequently the hypothesis of Picard's Theorem is not fulfilled by the DE in any region containing points on the x-axis.

(b) Note that $y \equiv 0$ is a solution of the IVP

$$\frac{dy}{dt} = |y| \qquad y(0) = 0$$

When $y < 0$, the DE becomes $y' = -y$, which has the general solution $y = Ce^{-t}$.

When $y \geq 0$, the DE becomes $y' = y$, which has general solution $y = Ce^{t}$.

Note that the only solution that satisfies the IVP occurs when $C = 0$, which is precisely the function $y \equiv 0$, so that is a unique solution.

■ Hubbard's Leaky Bucket

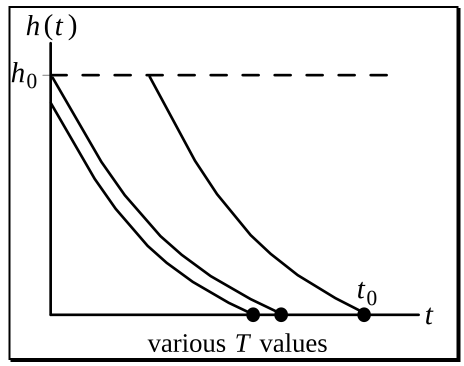
24. $\dfrac{dh}{dt} = -k\sqrt{h}$

(a) $f(t,\ h) = -k\sqrt{h}$, $\dfrac{\partial f}{\partial h} = -\dfrac{k}{2\sqrt{h}}$

Because $\dfrac{\partial f}{\partial h}$ is not continuous at $h = 0$, we cannot be sure of unique solutions passing through any points where $h(t) = 0$.

(b) Let us assume the bucket becomes empty at $t = T < t_0$. Solving the IVP with $h(T) = 0$, we find an infinite number of solutions.

$$h(t) = \frac{1}{4}(kT - kt)^2 \quad \text{for } t < T$$
$$h(t) = 0 \quad \text{for } t > T.$$



Each one of these functions describes the bucket emptying. Hence, we don't know when the bucket became empty. We show a few such solutions for $T < t_0$.

(c) If we start with a full bucket when $t = 0$, then (b) gives

$$h(0) = \frac{1}{4}k^2T^2 = h_0.$$

Hence the time to empty the bucket is

$$T = \frac{2}{k}\sqrt{h_0}.$$

■ The Melted Snowball

25. (a) We are given $\dfrac{dV}{dt} = -kA$, where A is the surface area of the snowball and $k > 0$ is the rate at which the snowball decreases in volume. Given the relationships between the volume of the snowball and its radius r, which is $V = \dfrac{4}{3}\pi r^3$, and between the surface area of the snowball and its radius, given by $A = 4\pi r^2$, we can relate A and V by

$$A = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} V^{2/3} = \sqrt[3]{36\pi}V^{2/3}.$$

(b) Here

$$f(t, V) = -kV^{2/3}$$
$$\frac{\partial f}{\partial V} = -\frac{2}{3}kV^{-1/3}.$$

Because the uniqueness condition for Picard's theorem does not hold when $V = 0$, we cannot conclude that the IVP

$$\frac{dV}{dt} = -kV^{2/3},\ V(t_0) = 0$$

has a unique solution. Hence, we cannot tell when the snowball melted; the backwards solution is not unique.

(c) Separating $\frac{dV}{dt} = -kV^{2/3}$ where $k > 0$, we have

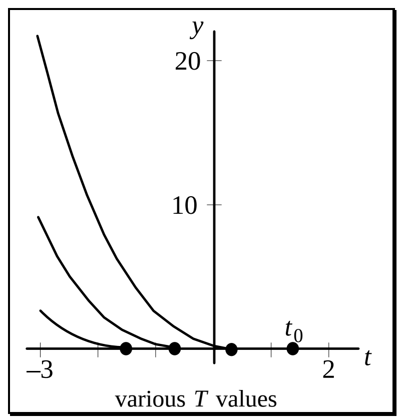
$$V^{-2/3}dV = -kdt.$$

Integrating, we find

$$3V^{1/3} = -kt + c.$$

Let $T < t_0$ be the time the snowball melted. Then using the initial condition $V(T) = 0$ we find

$$V(t) = -K\left(\frac{t-T}{3}\right)^3$$



$\frac{dV}{dt} = -kV^{2/3}$. Solutions with $y(t_0) = 0$.

for $K = k^3$ and $t < T$. But we know $V(t) \equiv 0$ is also is a solution of this initial-value problem, so we can piece together the nonzero solutions with the zero solution and get for $T < t_0$ the infinite family of solutions

$$V(t) = \begin{cases} -K\left(\frac{t-T}{3}\right)^3 & t < T \\ 0 & t \geq T. \end{cases}$$

(d) The function $f(t, V) = V^{2/3}$ does not satisfy the uniqueness condition of Picard's theorem when $V = 0$.

■ **The Accumulating Raindrop**

26. (a) We are given $\frac{dV}{dt} = kA$, where A is the surface area of the raindrop and $k > 0$ is the rate at which the raindrop increases in volume. We substitute into $\frac{dV}{dt} = kA$ the relationships

$$V = \frac{4}{3}\pi r^3,\ A = 4\pi r^2$$

for the volume *V* and area *A* of a raindrop in terms of its radius *r*, getting

$$A = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} V^{2/3} = \sqrt[3]{36\pi}V^{2/3}.$$

Hence

$$\frac{dV}{dt} = kV^{2/3}.$$

(b) Separating variables in the above DE, we have

$$V^{-2/3}dV = kdt.$$

Integrating, we find

$$3V^{1/3} = kt + c.$$

Using the initial condition $V(t_0) = 0$, we get the relation $c = -kt_0$, and hence

$$V(t) = K\left(\frac{t - t_0}{3}\right)^3$$

where $K = k^3$.

But clearly, $V(t) \equiv 0$ is also a solution of this initial-value problem, so we can piece together the nonzero solutions with the zero solution, to get the infinite family of solutions

$$V(t) = \begin{cases} 0 & t < t_0 \\ K\left(\frac{t - t_0}{3}\right)^3 & t \geq t_0 \end{cases}$$

■ Different Translations

27. (a) $y' = y$ has an infinite family of solution of the form $y = Ce^t$.

(To check: $y' = (Ce^t)' = Ce^t = y$.

Note that for any real number a,

$y = e^{t-a} = Ce^t$ is a solution for every $a \in R$.

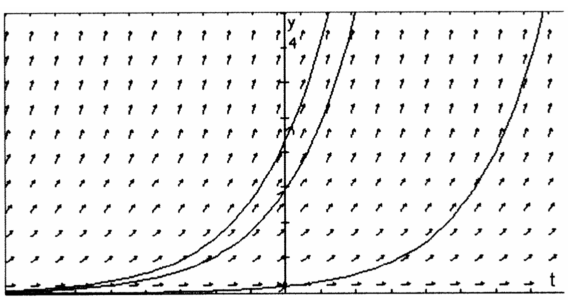
(b) Differentiating $s(t) = \begin{cases} 0 & t < a \\ (t-a)^2 & t \geq a \end{cases}$

we obtain a continuous derivative

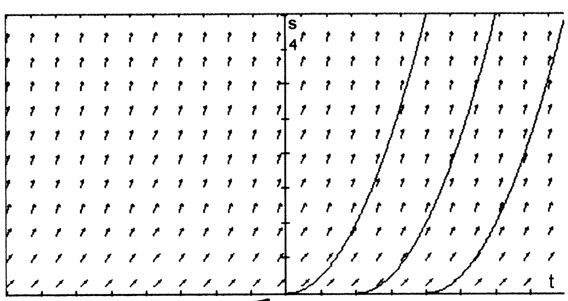
$$s'(t) = \begin{cases} 0 & t < a \\ 2(t-a) & t \geq a \end{cases}$$

Note that $s' = 2\sqrt{s}$ for both parts of the curve.

(c)



Some solutions for $y' = y$.



Some solutions for $s' = 2\sqrt{s}$.

For (a), with $y' = y$, solutions $y = Ce^t$ gradually approach zero as $t \to -\infty$.

For (b), with $s' = 2\sqrt{s}$, solutions $y = \begin{cases} 0 & \text{for } t < a \\ (t-a)^2 & \text{for } t \geq a \end{cases}$ go to zero at $t \to a$.

■ Picard Approximations

28.

$$y_0(t)=1$$

$$y_1(t)=1+\int_0^t (s-y_0)\,ds=1+\int_0^t (s-1)\,ds=1+\frac{1}{2}t^2-t$$

$$y_2(t)=1+\int_0^t \left\{s-\left(1+\frac{1}{2}s^2-s\right)\right\}ds=-\frac{1}{6}t^3+t^2-t+1$$

$$y_3(t)=1+\int_0^t \left\{s-\left[-\frac{1}{6}s^3+s^2-s+1\right]\right\}ds=\frac{1}{24}t^4-\frac{1}{3}t^3+t^2-t+1$$

29.

$$y_0(t)=t-1$$

$$y_1(t)=1+\int_0^t (s-y_0)\,ds=1+\int_0^t \left[s-(s-1)\right]ds=t+1$$

$$y_2(t)=1+\int_0^t \left[s-(s+1)\right]ds=-t+1$$

$$y_3(t)=1+\int_0^t \left[s-(1-s)\right]ds=t^2-t+1$$

30.

$$y_0(t)=e^{-t}$$

$$y_1(t)=1+\int_0^t (s-y_0)\,ds=1+\int_0^t \left(s-e^{-s}\right)ds=e^{-t}+\frac{1}{2}t^2$$

$$y_2(t)=1+\int_0^t \left[s-\left(e^{-s}+\frac{1}{2}s^2\right)\right]ds=e^{-t}-\frac{1}{6}t^3+\frac{1}{2}t^2$$

$$y_3(t)=1+\int_0^t \left[s-\left(e^{-s}-\frac{1}{6}s^3+\frac{1}{2}s^2\right)\right]ds=e^{-t}+\frac{1}{24}t^4-\frac{1}{6}t^3+\frac{1}{2}t^2$$

31.

$$y_0(t)=1+t$$

$$y_1(t)=1+\int_0^t (s-y_0)\,ds=1+\int_0^t \left[s-(1+s)\right]ds=1+\int_0^t -ds=1-t$$

$$y_2(t)=1+\int_0^t \left(s-(1-s)\right)ds=t^2-t+1$$

$$y_3(t)=1+\int_0^t \left[s-\left(s^2-s+1\right)\right]ds=-\frac{1}{3}t^3+t^2-t+1$$

■ Computer Lab

32. (a) We show how the computer algebra system Maple can be used to estimate the solution of #29 $y' = t - y$, $y(0) = 1$ with starting function $y_0(t) = t - 1$. We leave for the reader the other starting functions for #28, 30, and 31. In Maple open a new window and type the int() command. In this problem, because $f(t, y) = t - y$, $y(0) = 1$, we can find the sequence of approximations

$$y_{n+1}(t) = y_0 + \int_0^t f(s, y_n(s))\,ds = 1 + \int_0^t (s - y_n(s))\,ds$$

by typing

$$\begin{aligned}
y_0 &= t - 1;\\
y_1 &= 1 + \mathrm{int}(t - y_0, t);\\
y_2 &= 1 + \mathrm{int}(t - y_1, t);\\
y_3 &= 1 + \mathrm{int}(t - y_2, t);\\
y_4 &= 1 + \mathrm{int}(t - y_3, t);\\
y_5 &= 1 + \mathrm{int}(t - y_4, t);\\
y_6 &= 1 + \mathrm{int}(t - y_5, t).
\end{aligned}$$

If you then hit the enter key you will see displayed

$$\begin{aligned}
y_0 &= t - 1\\
y_1 &= t + 1\\
y_2 &= -t + 1\\
y_3 &= t^2 - t + 1\\
y_4 &= -\frac{1}{3}t^3 + t^2 - t + 1\\
y_5 &= \frac{1}{12}t^4 - \frac{1}{3}t^3 + t^2 - t + 1\\
y_6 &= -\frac{1}{60}t^5 + \frac{1}{12}t^4 - \frac{1}{3}t^3 + t^2 - t + 1
\end{aligned}$$

Of course you can find more iterates in the same way. E.g., if you type

$y_7 = 1 + \mathrm{int}(t - y_6, t)$,

then hit Enter, you will see

$$y_7 = -\frac{1}{360}t^6 - \frac{1}{60}t^5 + \frac{1}{12}t^4 - \frac{1}{3}t^3 + t^2 - t + 1$$

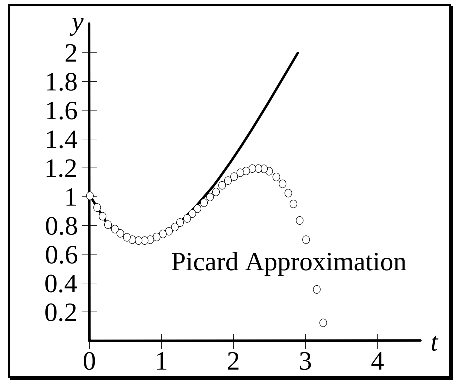
To get a plot of y_6 and the solution

$$y(t) = 2e^{-t} + t - 1$$

(see part (b)) as shown (see figure), type the Maple command

plot({2*exp(–*t*)+*t*–1, *y*6},

t=0..4, *y*=0..2).



Picard's sixth approximation to

$$y' = t - y,\ y(0) = 1$$

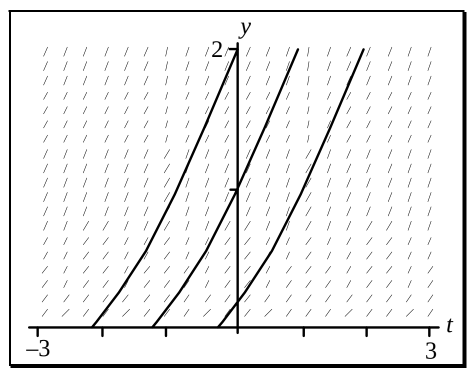
(b) If you recall the Maclaurin series $e^{-2t} \approx 1 - 2t + \frac{1}{2}(-2t)^2 + \frac{1}{3!}(-2t)^3 + \dots$ and carry out a little algebra, you will convince yourself that these Picard's approximations are converging to the analytical solution $y(t) = 2e^{-t} + t - 1$. For most initial-value problems, however, such a nice identification is not possible.

■ **Calculator or Computer**

33. $y' = f(t,\ y) = y^{1/4}$

$$\frac{\partial f}{\partial y} = \frac{1}{4} y^{-3/4}$$

Note the direction field is only defined when $y \ge 0$. Picard's theorem guarantees existence through any point $y(t_0) = y_0$, but not uniqueness for points $y(t_0) = y_0$ when $y_0 = 0$. The direction field shown illustrates these ideas.

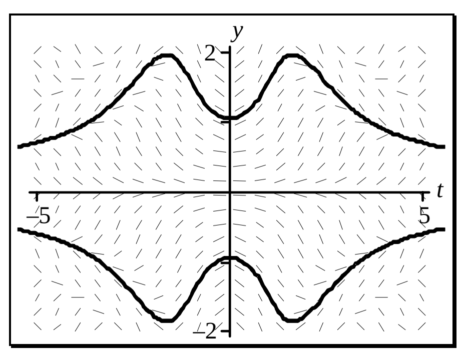


$y' = y^{1/4}$; DE does not exist for $y < 0$.

34. $y' = f(t,\ y) = \sin(ty)$

$$\frac{\partial f}{\partial y} = t\cos(ty)$$

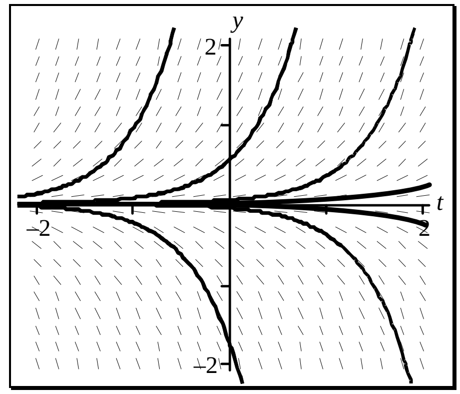
Picard's theorem guarantees both existence and uniqueness for any point $(t_0,\ y_0)$. The direction field shown also indicates these ideas.



35. $y' = f(t, y) = y^{5/3}$

$$\frac{\partial f}{\partial y} = \frac{5}{3} y^{2/3}$$

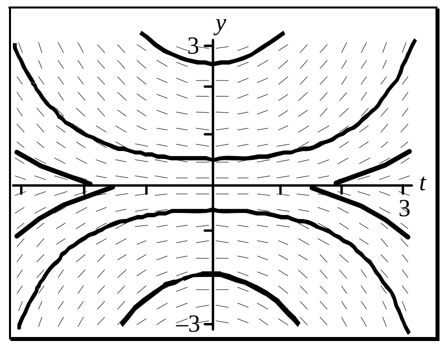
Picard's theorem guarantees existence and uniqueness for all initial conditions $y(t_0) = y_0$. The direction field shown also illustrates this fact.



36. $y' = f(t, y) = (ty)^{1/3}$

$$\frac{\partial f}{\partial y} = \frac{1}{3}(ty)^{-2/3} t = \frac{1}{3} t^{1/3} y^{-2/3}$$

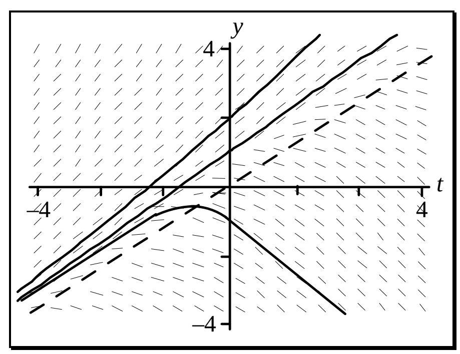
The function f is continuous for all (t, y), but f_y is not continuous when $y = 0$. Hence, we are not guaranteed uniqueness through points (t_0, y_0) when y_0 is zero. See figure for this direction field.



37. $y' = f(t, y) = (y - t)^{1/3}$

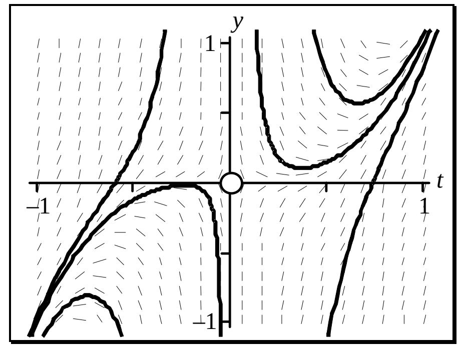
$$\frac{\partial f}{\partial y} = \frac{1}{3}(y - t)^{-2/3}$$

The function f is continuous for all (t, y), but f_y is not continuous when $y = t$ (it doesn't exist). Hence, the DE has a solution through every point (t_0, y_0) but Picard's theorem does not guarantee uniqueness through points for which $y_0 = t_0$. See figure for this direction field.



38. $$y' = f(t,\ y) = 6t^2 - \frac{3}{t}y$$

$$\frac{\partial f}{\partial y} = -\frac{3}{t}$$



The function f is continuous except when $t = 0$, hence there exists a solution through all points $(t_0,\ y_0)$ except possibly when $t_0 = 0$. Also f_y is continuous except when $t = 0$, and so the DE has a unique solution for all initial conditions except possibly when $t_0 = 0$. The direction field of the equation as shown indicates that no solutions pass through initial points of the form $(0,\ y_0)$.

■ **Suggested Journal Entry**

39. Student Project

CAS Project 1A Direction Fields

In this project you will use your CAS to create direction fields for first order differential equations. Solution curves can be added if your CAS provides this feature. Using `Maple` the relevant commands are

> **DEtools**
> A package of procedures that includes commands for plotting direction fields. Load the procedures using the entry **with(DEtools)**.
>
> **DEplot**
> A command in the **DEtools** package that plots direction fields and numerically generated solution curves.
>
> **plots**
> A package of procedures that create specialized plots of one sort or another. Load the procedures using the entry **with(plots)**.
>
> **display**
> The **display** command is used to display several plots in one picture. It is in the **plots** package.
>
> **implicitplot**
> Another command in the **plots** package, **implicitplot** draws curves that are defined implicitly.

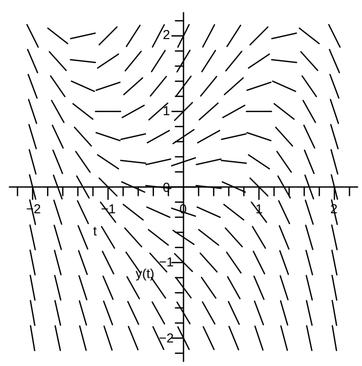
Example. For the differential equation $y' = y - t^2$ plot a direction field. Use the window $-2 \le t \le 2$, $-2 \le y \le 2$ with 13 slope marks in each direction.

The first entry below loads the **DEtools** package. We recommend that you define and name the differential equation first, just to make sure that it has been entered correctly.

```
> with(DEtools):                                      #Load the DEtools package.
> DE := diff(y(t),t) = y(t) - t^2;                    #Name the DE.
```

$$DE := \frac{d}{dt}y(t) = y(t) - t^2$$

```
> DEplot( DE, y(t), t=-2..2, y=-2..2, arrows=line,    #Plot the direction field.
          dirgrid=[13,13], scaling=constrained);
```



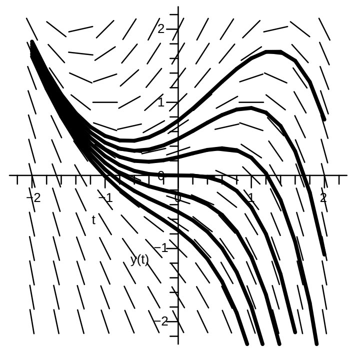
The direction field for $y' = y - t^2$.

Things for you to do.

1. Print the plot displayed above and use the slope marks to draw several solution curves by hand.
2. Use your CAS to add some solution curves to the plot and compare them to your sketched curves. The `Maple` code for adding solution curves starting at $y(0) = 0.25k$ for $k = -3 \ldots 3$ is shown below. The curves are generated numerically (i.e. without a solution formula) using the celebrated Runge-Kutta algorithm as described in Section 1.4. The first entry, named **inits**, defines the set of initial values for

the solution curves. For each initial condition Maple plots 20 equally spaced points from left to right across the slope field and connects them with straight lines.

```
> inits := { [y(0)=0.25*k] $ k=-3..3 }:                  #Name the set of inits.
> DEplot( DE, y(t), t=-2..2, y=-2..2, arrows=line,       #Plot the solution curves.
          dirgrid=[13,13], scaling=constrained,
          inits, linecolor=black);
```



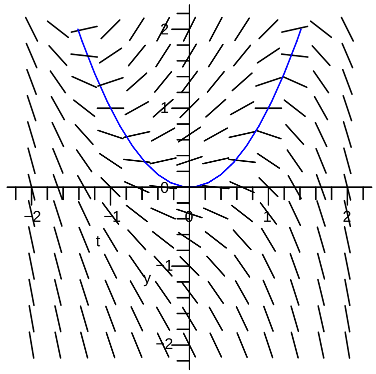
The direction field and several solution curves for $y' = y - t^2$.

3. Obtain the direction field for the DE using the window $-4 \leq t \leq 4$, $-4 \leq y \leq 4$. Ask for 17 tangent lines in each direction with the equation **dirgrid = [17,17]**.
   a) Print the field and draw several solution curves by hand.
   b) Use your CAS to draw the solution curves starting at $y(0) = 0.5k$ for $k = -6 \ldots 6$. Note how the solutions change as the initial values increase.
   c) This differential equation is not separable, but it is linear and you will learn how to solve it in the next chapter. Use the general solution formula: $y(x) = t^2 + 2t + 2 + Ce^t$, to explain why the solution curves change so abruptly as the initial value of $y(0)$ increases.
4. For the DE $y' = y^2 - t$ draw the direction field. Include a plot of the nullcline. Comment on the relationship between the nullcline and the field of tangent lines.

   The Maple code for the DE $y' = y - t^2$ is shown below. The plot of the nullcline uses the **implicitplot** command. The **display** command is used to show the nullcline and the direction field together.

> Maple allows you to differentiate between curves by plotting them in different colors. Although printing constraints prevent us from showing them in color in the text, we will use color in some of our examples. We will point out a curve and show the expected color in parentheses. See the example below.

```
> with(plots):                                            #Load the plots package.
> NullCline := implicitplot( y-t^2=0, t=-2..2, y=-2..2, thickness=1,
                             color=blue, scaling=constrained):
  display( NullCline,                                     #Display the nullcline plot
    DEplot( DE, y(t), t=-2..2, y=-2..2, arrows=line,      #and the direction field.
    dirgrid=[13,13], color=black, scaling=constrained) );
```



The direction field and its nullcline (blue) for $y' = y - t^2$.

5. Analyze the solution curves for the differential equation $y' = y^2 - t^2$ using its direction field. Sketch some solution curves by hand, then use your CAS to plot several numerically-generated solution curves. Also make a plot showing the direction field and the nullclines.

CAS Project 1B Comparing Numerical Methods

In this project you will use your CAS to generate approximate solutions to an initial value problem (IVP) using the Euler one-step algorithm. The midpoint Euler algorithm (second-order Runge-Kutta) is also implemented. Using `Maple` the relevant concepts and commands are

> **User defined tables**
> The symbol **Y[k]** is referred to as a subscripted variable; **k** is the subscript. When such a symbol first appears in a `Maple` worksheet a table named **Y** is created in memory and can be used to store data of almost any sort. For example, the entry **Y[3] := 23** stores the number 23 in the "3-position" in the table **Y**.
>
> **for..do loops**
> Simple looping constructs can be made easily. The data that is created in a **for..do loop** can be stored in a table. The syntax is self-explanatory.
>
> **dsolve**
> The **dsolve** command solves differential equations symbolically. The entry **dsolve(DE)** where **DE** represents the differential equation, outputs the general solution. The entry **dsolve({DE, y(0)=y0})** outputs the solution equation for an IVP.
>
> **unapply**
> The **unapply** command is used to convert an expression into a function. For example, if an expression in the variable t has the name "Joe", it can be made into a function named "f" with the entry **f := unapply(Joe, t)**. Thereafter, $f(t) = Joe$.

Example. Use the Euler one-step algorithm to approximate the solution values for the IVP $y' = y - t^2$, $y(0) = 0.5$ for $t = 0.2, 0.4, \ldots, 2.0$. Plot the points, then add the actual solution curve.

The first entry loads the **plots** package for ready access to the **display** procedure. The second entry defines the function f so that the differential equation is $y' = f(t, y)$. As usual, we recommend that you define and name the differential equation, just to make sure that it has been entered correctly. The rest of the entries are annotated.

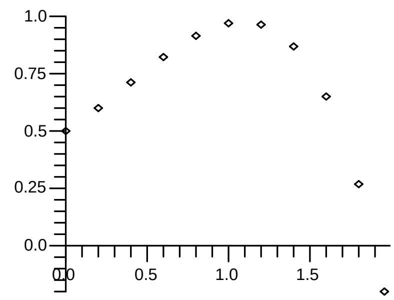
```
> with(plots):
> f := (t,y) -> y - t^2:
  DE := diff(y(t),t) = f(t,y(t));
```

$$DE := \frac{d}{dt}y(t) = y(t) - t^2$$

```
> T[0] := 0: Y[0] := 0.5: h := 0.2:        #Define tables T and Y, step size h.
  for n from 0 to 9                         #The for..do loop starts here.
    do
      T[n+1] := T[n] + h:
      Y[n+1] := Y[n] + h*f(T[n],Y[n]):
    end do:                                 #The loop ends here.
  unassign('n');                            #The looping index is unassigned for later use.
> Matrix( [ ['n', 0.2*k $ k=0..8],          #A matrix is created to display some
            ['y[n]', Y[n] $ n=0..8] ] ):    #approximate solution values.
  evalf[3](%);                              #Evaluate to 3-digit accuracy the line above.
```

$$\begin{bmatrix} n & 0. & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 & 1.2 & 1.4 & 1.6 \\ y_n & 0.5 & 0.60 & 0.712 & 0.822 & 0.915 & 0.970 & 0.964 & 0.869 & 0.650 \end{bmatrix}$$

```
> plot( [ [n*h,Y[n]] $ n=0..10], view=[0..2,-0.2..1],
          style=point, symbolsize=16);      #Create a plot of the approximation points.
  Approx := %:                              #Store with the name "Approx" the plot output.
```

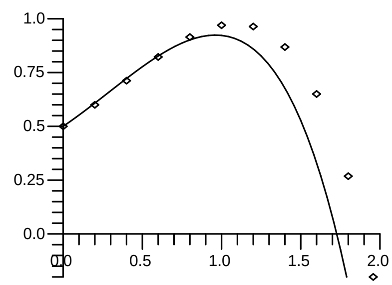


Euler one-step approximations to the solution curve in the ty plane.

```
> soln := dsolve( {DE,y(0)=0.5} ):          #Obtain the solution formula.
  g := unapply(rhs(soln),t):                #Define the solution function g.
  'g(t)' = g(t);                            #Show the formula for g(t).
```

$$g(t) = 2 + 2t + t^2 - \frac{3}{2}e^t$$

```
> display( Approx,                          #Display approximations and solution curve.
           plot( g(t), t=0..2, color=black)
         );
```



The solution curve and the Euler one-step approximations.

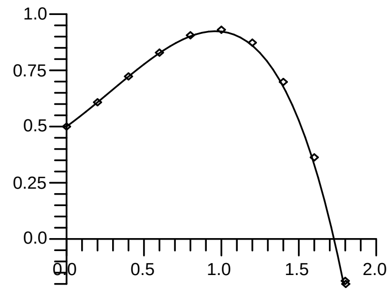
Things for you to do.

1. a) Change the stepsize to 0.05 for Euler's method and compare the results with the example.

   b) Alter the code to implement the midpoint Euler algorithm. This is easy, simply change the for..do loop to look like this. (See Section 1.4 in the text).

```
for n from 0 to 9
   do
     T[n+1] := T[n] + h:
     kn1 := f(T[n],Y[n]):
     kn2 := f(T[n]+h/2,Y[n]+h/2*kn1):
     Y[n+1] := Y[n] + h*kn2:
   end do:
```

   The plot of the midpoint approximation points and the solution curve shows a dramatic increase in accuracy. Check your code by verifying that it produces the following picture.



The solution curve and the midpoint Euler approximation points.

2. For the IVP $y' = -t/y$, $y(0) = 1$, obtain similar pictures showing the Euler one-step and the midpoint Euler approximations for $t = 0.2, 0.4, 0.6, 0.8, 1.0$. The solution curve is a circle. Find its equation, and add the solution curve to your plots.

3. Study the famous fourth-order Runge-Kutta algorithm described at the end of Section 1.4. Modify the code above to implement this algorithm. Test it on the given IVP and the IVP described in part 2.

4. For $y' = y$, $y(0) = 1$, try each one of the methods, Euler, midpoint Euler, and fourth order Runge-Kutta, on $[0, 2]$ with step size 0.5, and compare the results.