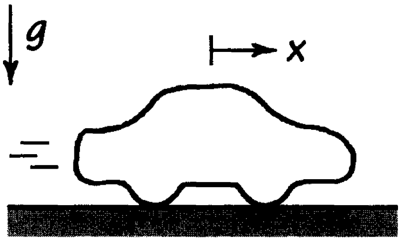
2.1.1

GOAL: Determine the distance and time needed for the car to reach its maximum speed.

GIVEN: The 2007 BMW Z4 Coupe 3.0si can accelerate from 0 to 96 km/hr in 5.6 s, and its maximum speed is $\dot{x}_{\text{max}} = 249\,\text{km/hr}$. Assume that it accelerates from 0 to 96 km/hr at a constant rate and that this acceleration is maintained as the vehicle pushes toward its maximum speed.

DRAW:



FORMULATE EQUATIONS:

Since we're assuming that the car's acceleration is constant, we can say that

$$\ddot{x} = \frac{\Delta\dot{x}}{\Delta t} \tag{1}$$

$$\dot{x}^2 - \dot{x}_0^2 = 2\ddot{x}\Delta x \tag{2}$$

SOLVE:

To go from 0 to 96 km/hr in 5.6 s at a constant rate, the car's acceleration needs to be

(1) $\Rightarrow$

$$\ddot{x} = \frac{(96\,\text{km/hr})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{10^3\,\text{m}}{\text{km}}\right)}{5.6\,\text{s}}$$
$$\ddot{x} = 4.76\,\text{m/s}^2$$

If the car continues to accelerate at this rate, then it will reach its maximum speed after traveling

(2) $\Rightarrow$

$$\Delta x_{\text{max}} = \frac{(\dot{x}_{\text{max}})^2}{2\ddot{x}}$$

$$\Delta x_{\text{max}} = \frac{\left[(249\,\text{km/hr})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{10^3\,\text{m}}{\text{km}}\right)\right]^2}{2(4.76\,\text{m/s}^2)}$$

$$\boxed{\Delta x_{\text{max}} = 502.5\text{m} = 0.5025\text{km}}$$

The time it takes for the car to attain maximum speed is

(1) $\Rightarrow$
$$\Delta t_{\text{max}} = \frac{\dot{x}_{\text{max}}}{\ddot{x}}$$

$$\Delta t_{\text{max}} = \frac{(249\,\text{km/hr})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{10^3\,\text{m}}{\text{km}}\right)}{4.76\,\text{m/s}^2}$$

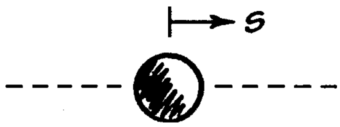
$$\boxed{\Delta t_{\text{max}} = 14.5\,\text{s}}$$

2.1.5

GOAL: Determine the ball bearing's speed v after traveling the given distance.

GIVEN: The ball bearing's acceleration is described by $\ddot{s} = as + be^{cs}$, where $a = 3\,\mathrm{s}^{-2}$, $b = 0.3\,\mathrm{m/s^2}$, and $c = 0.06\,\mathrm{m}^{-1}$. The ball bearing travels $d = 3\,\mathrm{m}$, and it starts from rest.

DRAW:



TOP VIEW

FORMULATE EQUATIONS:

Since the ball bearing's acceleration is a function of its position,

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{d\dot{s}}{ds} \cdot \frac{ds}{dt} = \frac{d\dot{s}}{ds}\dot{s}$$

$$\ddot{s}ds = \dot{s}d\dot{s} \tag{1}$$

SOLVE:

(1) $\Rightarrow$

$$(as + be^{cs})\,ds = \dot{s}d\dot{s}$$

$$\int_0^d (as + be^{cs})\,ds = \int_0^v \dot{s}d\dot{s}$$

$$\left(\frac{1}{2}as^2 + \frac{b}{c}e^{cs}\right) \int 0d = \frac{1}{2}\dot{s}^2 \int 0v$$

$$\frac{1}{2}ad^2 + \frac{b}{c}\left(e^{cd} - 1\right) = \frac{1}{2}v^2$$

$$v = \sqrt{ad^2 + \frac{2b}{c}\left(e^{cd} - 1\right)}$$

$$v = \sqrt{(3s^{-2})(3\,\mathrm{m})^2 + \frac{2(0.3\,\mathrm{m/s^2})}{0.06\,\mathrm{m}^{-1}}(e^{(0.06\,\mathrm{m}^{-1})(3\,\mathrm{m})} - 1)}$$

$$\boxed{v = 5.38\,\mathrm{m/s} = 19.37\mathrm{km/h}}$$

2.1.6

GOAL: Find the constant acceleration a_0 that brings a jet from 274 km/hr to 0 km/h in 73m and the elapsed time Δt.

GIVEN: Distance needed to go from landing speed to zero.

FORMULATE EQUATIONS: Because the acceleration is constant we can use

$$v_2^2 - v_1^2 = 2a_0(x_2 - x_1) \tag{1}$$

$$v_2 - v_1 = a_0 \Delta t \tag{2}$$

where a_0 is the constant acceleration, Δt denotes the elapsed time, and the subscripts 1,2 indicate initial and final conditions, respectively.

SOLVE: First we'll convert 274 km/h to m/s:

$$\frac{(274\text{km})}{(1\text{hr})} \times \frac{(10^3\text{m})}{(1\text{km})} \times \frac{(1\text{hr})}{(3600\text{s})} = 76.1\text{m/s}$$

Using this, along with the known distance traveled, in (1) gives us

$$0 - (76.1\text{m/s})^2 = 2\text{a}_0(73\text{m})$$

$$\boxed{\text{a}_0 = -39.7\text{m/s}^2}$$

Because 1 g is equal to 9.81 m/s^2 we have

$$\boxed{a_0 = -39.7\,\text{m/s}^2 \frac{(1\,g)}{(9.81\,\text{m/s}^2)} = -4.05\,g}$$

We can then use (2) to find the time Δt taken for the jet to come to a halt:

$$0 - (76.1\text{m/s}) = (-39.7\text{m/s}^2 \Delta t)$$

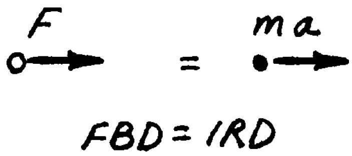
$$\boxed{\Delta t = 1.92\,\text{s}}$$

2.1.7

GOAL: Find the needed deck space to allow a fighter jet to take off from an aircraft carrier.

GIVEN: The jet is brought from 0 to 265 km/h in 2.23 s.

DRAW:



ASSUME: We'll assume that the acceleration is constant.

FORMULATE EQUATIONS: We'll use the equation relating speed and acceleration, when starting from rest under a constant acceleration:

$$v = at$$

and the expression for distance traveled under a constant acceleration:

$$x = \frac{1}{2}at^2$$

SOLVE:

First we'll convert 265 km/h to m/s:

$$\frac{(265 \text{ km})}{(1 \text{ hr})} \times \frac{(10^3 \text{ m})}{(1 \text{ km})} \times \frac{(1 \text{ hr})}{(3600 \text{ s})} = 73.6 \text{ m/s}$$

The acceleration is therefore

$$a = \frac{v}{t} = \frac{73.6 \text{ m/s}}{2.23 \text{ s}} = 33 \text{ m/s}^2$$

The needed space is given by

$$x = \frac{1}{2}(33 \text{ m/s}^2)(2.23 \text{ s})^2$$

$$\boxed{x = 82 \text{ m}}$$

2.1.9

GOAL: Find the speed, $\dot{x}$, and the acceleration, $\ddot{x}$, at prescribed times.

GIVEN: x as a function of time.

GOVERNING EQUATIONS: The position of the car is given by:

$$x(t) = 30\ \mathrm{m} - (27\ \mathrm{m/s})t + (3\ \mathrm{m/s^2})t^2 \tag{1}$$

Differentiating with respect to time, we get:

$$\dot{x}(t) = -27\ \mathrm{m/s} + (6\ \mathrm{m/s^2})t \tag{2}$$

and

$$\ddot{x}(t) = 6\ \mathrm{m/s^2} \tag{3}$$

SOLVE:

(2) $\Rightarrow$

$$\boxed{\dot{x}(2) = -27\ \mathrm{m/s} + (6\ \mathrm{m/s^2})(25) = -15\ \mathrm{m/s}} \tag{4}$$

and

(3) $\Rightarrow$

$$\boxed{\ddot{x}(10) = 6\ \mathrm{m/s^2}} \tag{5}$$

2.1.10

GOAL: Find the constant acceleration a_0 that brings a car from 0 to 96 km/h in 5 seconds.

GIVEN: Time needed to go from zero to 96 km/h.

FORMULATE EQUATIONS: Because the acceleration is constant we have

$$v(t) = v(0) + a_0 t \tag{1}$$

where a_0 is a constant acceleration.

SOLVE: First we'll convert 96 km/h to m/s:

$$\frac{(96 \text{ km})}{(1 \text{ hr})} \times \frac{(10^3 \text{ m})}{(1 \text{ km})} \times \frac{(1 \text{ hr})}{(3600 \text{ s})} = 26.7 \text{ m/s}$$

Using this in (1) gives us

$$26.7 \text{ m/s} = 0 + \text{ a}_0(5 \text{ s})$$

$$\boxed{a_0 = 5.3 \text{ m/s}^2}$$

Because 1 g is equal to 9.81 m/s^2 we have

$$\boxed{a_0 = 5.3 \text{ m/s}^2 \frac{(1\,g)}{(9.81 \text{ m/s}^2)} = 0.54\,g}$$

2.1.11

GOAL: Determine the time to bring a car to a stop from an initial speed along with the distance over which stopping occurs.

GIVEN: Initial speed of car and the fact that the car decelerates at 1 g.

FORMULATE EQUATIONS:

For a constant acceleration,

$$v(t) = v(0) + at$$

$$x(t) = x(0) + v(0)t + \frac{at^2}{2}$$

SOLVE:

$$113 \text{ km/h} = \frac{113 \times 10^3}{8600} \text{ m/s} = 31.4 \text{ m/s}$$

$$v(t^*) = 31.4 \text{ m/s} - (9.81 \text{ m/s}^2)t^* = 0 \Rightarrow \boxed{t^* = 3.2\,\text{s}}$$

And for the distance:

$$x(t^*) = 0 + (31.4 \text{ m/s})t^* - \frac{9.81 \text{ m/s}^2}{2} t^{*2}$$

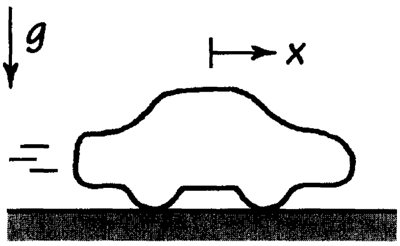
$$\boxed{\text{At } t^* = 3.2 \text{ s}, \ x = 50.3 \text{ m}}$$

2.1.12

GOAL: Determine the distance the car needs to reach its maximum speed and the change in its 0-to-60 time such that it and a competitor's car reach their respective maximum speed in the same distance.

GIVEN: The 2008 Audi TT Coupe can accelerate from 0 to 96 km/hr in 6.5 s and has a maximum speed of $\dot{x}_{\text{max}} = 237\,\text{km/hr}$. Assume that it accelerates from 0 to 96 km/hr at a constant rate and that this acceleration is maintained as the vehicle pushes toward its maximum speed.

DRAW:



FORMULATE EQUATIONS:

Since we're assuming that the car's acceleration is constant, we can say that

$$\ddot{x} = \frac{\Delta \dot{x}}{\Delta t} \tag{1}$$

$$\dot{x}^2 - \dot{x}_0^2 = 2\ddot{x}\Delta x \tag{2}$$

SOLVE:

To go from 0 to 96 km/h in 6.5 s at a constant rate, the car's acceleration needs to be

(1) $\Rightarrow$

$$\ddot{x} = \frac{(96\ \text{km/hr})\left(\frac{\text{hr}}{3600\ \text{s}}\right)\left(\frac{10^3\ \text{m}}{\text{km}}\right)}{6.5\ \text{s}}$$

$$\ddot{x} = 4.1\ \text{m/s}^2$$

If the car continues to accelerate at this rate, then it will reach its maximum speed after traveling

(2) $\Rightarrow$

$$\Delta x_{\text{max}} = \frac{(\dot{x}_{\text{max}})^2}{2\ddot{x}}$$

$$\frac{\left[(237\text{ km/hr})\left(\frac{\text{hr}}{3600\text{ s}}\right)\left(\frac{10^3\text{ m}}{\text{km}}\right)\right]^2}{2(4.1\text{ m/s}^2)}$$

$$\boxed{\Delta x_{\text{max}} = 529\text{ m} = 0.529\text{ km}}$$

We find that it takes a longer distance for the TT to reach its maximum speed as compared to its competitor. For the car to attain maximum speed in 0.50 km like its competitor, its 0-to-60 time would need to be

(1) $\rightarrow$ (2) $\Rightarrow$

$$(\dot{x}_{\text{max}})^2 = \frac{2(\Delta\dot{x})(\Delta x)}{t^*}$$

$$t^* = \frac{2(\Delta\dot{x})(\Delta x)}{(\dot{x}_{\text{max}})^2}$$

$$t^* = \frac{2(96\text{ km/hr})\left(\frac{\text{hr}}{3600\text{ s}}\right)\left(\frac{10^3\text{ m}}{\text{km}}\right)(0.5\text{ km})\left(\frac{10^3\text{ m}}{\text{km}}\right)}{\left[(237\text{ km/hr})\left(\frac{\text{hr}}{3600s}\right)\left(\frac{10^3\text{ m}}{\text{km}}\right)\right]^2}$$

$$t^* = 6.15\,\text{s}$$

Thus, the car's 0-to-60 time would need to decrease by

$$\Delta t = 6.5\,\text{s} - 6.15\,\text{s}$$

$$\boxed{\Delta t = 0.35\,\text{s}}$$

2.1.13

GOAL: Determine a piston's maximum speed and acceleration.

GIVEN: Position of piston as a function of time.

FORMULATE EQUATIONS: The governing equation of motion is given as

$$x(t) = \frac{8.97}{2} \sin \omega t$$

where x is given in cm and $\omega = 7000$ rpm $= 733$ rad/s.

SOLVE:

$$v(t) = \frac{d}{dt} x(t) = \left(\frac{8.97 \text{ cm}}{2} \right) \omega \cos \omega t$$

$$\boxed{v_{max} = \left(\tfrac{8.97 \text{ cm}}{2}\right) (733 \text{ rad/s}) = 3.29 \times 10^3 \text{ cm/s}}$$

$$a(t) = \frac{d}{dt} v(t) = -\left(\frac{8.97 \text{ cm}}{2} \right) \omega^2 \sin \omega t$$

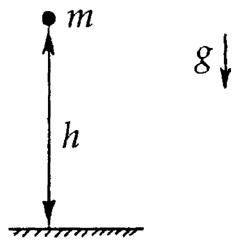
$$\boxed{a_{max} = \left(\tfrac{8.97 \text{ cm}}{2}\right) (733 \text{ rad/s})^2 = 2.4 \times 10^6 \text{ cm/s}^2}$$

2.1.14

GOAL: Find the height h for which a falling body will contact the ground at 56 km/hr.

GIVEN: Speed of contact.

DRAW:



ASSUME:

$$\begin{aligned} v_i &= 0 \\ v_f &= 56\,\text{km/hr} = 15.6\,\text{m/s} \\ a &= 9.81\,\text{m/s}^2 \end{aligned} \tag{1}$$

FORMULATE EQUATIONS: We'll use the formula for the difference in speed due to a constant acceleration a over a distance h:

$$v_f^2 - v_i^2 = 2ah \tag{2}$$

SOLVE:

(1)→(2)⇒

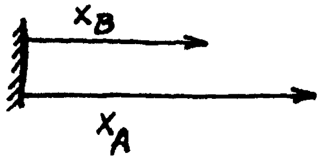
$$\begin{aligned} v_f^2 - v_i^2 &= 2gh \\ h &= \frac{v_f^2 - v_i^2}{2g} = \frac{(15.6\,\text{m/s})^2 - 0}{2(9.81\,\text{m/s}^2)} \\ h &= \boxed{12.4 \text{ m}} \end{aligned}$$

2.1.16

GOAL: Find the constant speed needed for a pursuing cyclist to catch another cyclist.

GIVEN: Initial positions of the cyclists and the lead cyclist's speed.

DRAW:



FORMULATE EQUATIONS: We'll need to use the formula for position as a function of time due to a constant speed v:

$$x(t) = x(0) + vt$$

SOLVE: Bicyclist A has to travel 1610 m at the speed of 29 km/h. Thus we have

$$1610 \text{ m} = \left[\frac{(29 \times 10^3)}{3600} \text{ m/s}\right] t \Rightarrow t = 200 \text{ s}$$

Bicyclist B has to travel an additional 229 m, for a total of 1839 m and has 200 s to do so. Thus we have

$$1839 \text{ m} = v_B(200 \text{ s})$$

$$\boxed{v_B = 9.2 \text{ m/s} = 33.12 \text{ km/hr}}$$

2.1.17

GOAL: Find the terminal speed of an object with a given acceleration and determine at what time it reaches 95 percent of terminal speed.

GIVEN: $a_0 = 24,384\,\mathrm{m/s^2}$, $a_1 = 0.108\,\mathrm{s/\,m^2}$

FORMULATE EQUATIONS: The acceleration is given by

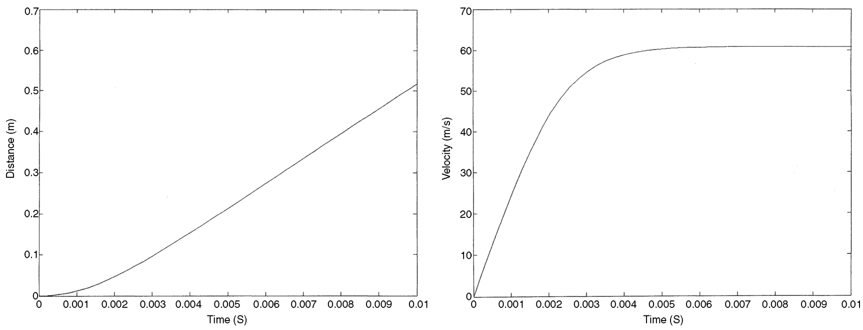
$$\ddot{s} = a_0 - a_1 \dot{s}^3$$

SOLVE:

We can numerically integrate using MATLAB with initial conditions of $s(0) = 0$, $\dot{s}(0) = 0$. Using a time interval from $t = 0$ to $t = 0.01\,\mathrm{s}$ yields the following plot:

It can be seen from the plot that the terminal speed is

$$\boxed{v_{term} = 60.9\,\mathrm{m/s}}$$



(a) (b)

This result can be seen analytically as well. When v_{term} is reached, the acceleration $\ddot{s}$ is zero. Using this in our acceleration equation gives

$$0 = a_0 - a_1 v_{term}^3$$

which, when solved, returns the result $v_{term} = 60.9\,\mathrm{m/s}$.

An examination of the output data allows the time at which the speed reaches 95 percent of its terminal value $(0.95(60.9\,\mathrm{m/s}) = 57.9\,\mathrm{m/s})$ to be determined as

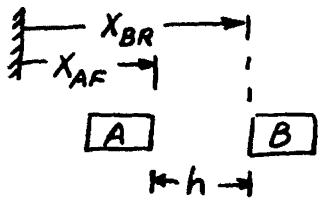
$$\boxed{t_{term} = 0.00367\,\mathrm{s}}$$

2.1.22

GOAL: Find the time of collision between Car A to hit Car B and their relative speed at collision.

GIVEN: At $t = 0\,\mathrm{s}$ Car A is traveling at a constant speed of 30 m/s and Car B is 6.9 m in front of Car A, traveling at 24 m/s and decelerating at 6 m/s^2. At $t = 0.5\,\mathrm{s}$ Car A decelerates at a constant 9 m/s^2. The separation of the two cars is given by h.

DRAW:



FORMULATE EQUATIONS: We'll use the general expressions for position and speed, given a constant acceleration and initial position and speed x_0, v_0:

$$x(t) = x_0 + v_0 t + \frac{at^2}{2}$$

$$v(t) = v_0 + at$$

SOLVE:

At $t = 0\,\mathrm{s}$ let the position of Car A be $x_{AF} = 0\,\mathrm{m}$ and the position of Car B be $x_{BR} = 6.9\,\mathrm{m}$. After $0.5\,\mathrm{s}$ have elapsed the positions and speeds are found from

$$x_{AF}(0.5\,\mathrm{s}) = (30\,\mathrm{m/s})(0.5\,\mathrm{s}) = 15\,\mathrm{m}$$

$$x_{BR}(0.5\,\mathrm{s}) = 6.9\,\mathrm{m} + (24\,\mathrm{m/s})(0.5\,\mathrm{s}) - 0.5(6\,\mathrm{m/s^2})(0.5\,\mathrm{s})^2 = 18.9\,\mathrm{m} - 0.75\,\mathrm{m} = 18.15\,\mathrm{m}$$

$$v_{AF}(0.5\,\mathrm{s}) = 30\,\mathrm{m/s}$$

$$v_{BR}(0.5\,\mathrm{s}) = 24\,\mathrm{m/s} - (6\,\mathrm{m/s^2})(0.5s) = 21\,\mathrm{m/s}$$

At this point Car A begins to decelerate at $-9\,\mathrm{m/s^2}$. For convenience we'll reset time to zero (t indicates time beyond the $0.5\,\mathrm{s}$ needed for Car A's braking to begin.)

$$x_{AF}(t) = 15\,\mathrm{m} + (30\,\mathrm{m/s})t + 0.5(-9\,\mathrm{m/s^2})t^2$$

$$x_{BR}(t) = 18.15\,\mathrm{m} + (21\,\mathrm{m/s})t + 0.5(-6\,\mathrm{m/s^2})t^2$$

The separation h is given by

$$h = x_{BR} - x_{AF} = 3.15\,\mathrm{m} - (9\,\mathrm{m/s})t + (1.5\,\mathrm{m/s^2})t^2$$

and the collision occurs when $h = 0$. We're left with a quadratic to solve:

$$3.15\,\mathrm{m} - (9\,\mathrm{m/s})t + (1.5\,\mathrm{m/s^2})t^2 = 0$$

$$t^2 - (6\,\mathrm{s})t + 2.1\,\mathrm{s}^2 = 0$$

The relevant solution to this equation is $t = 0.373\,\mathrm{s}$. Thus the elapsed time from when the driver of Car A first decides to brake is given by $0.5\,\mathrm{s} + 0.373\,\mathrm{s} = \boxed{0.873\,\mathrm{s}}$.

The collision speed is found from

$$-\dot{h} = (-21\,\mathrm{m/s} + 30\,\mathrm{m/s}) + (-9\,\mathrm{m/s^2} + 6\,\mathrm{m/s^2})(0.354\,\mathrm{s}) = 7.94\,\mathrm{m/s}$$

Thus we have a collision speed of

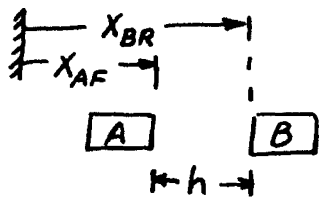
$$\boxed{7.94\,\mathrm{m/s} = 28.58\,\mathrm{km/h}}$$

2.1.23

GOAL: Determine whether Car A hits Car B and, if so, determine the relative speed of the collision.

GIVEN: Initially both cars are traveling at 30 m/s. Car A is initially 9 m behind Car B. Car B begins to decelerate at $10.7\,\mathrm{m/s^2}$ and 1 s later Car A begins to decelerate at $9\,\mathrm{m/s^2}$.

DRAW:



FORMULATE EQUATIONS:

We'll use the formulas for motion with constant acceleration a (initial position and speed given by x_0, v_0):

$$v(t) = v_0 + at$$

$$x(t) = x_0 + v_0 t + \frac{at^2}{2}$$

SOLVE:

x_{AF} indicates the front position of Car A and x_{BR} indicates the position of the rear of Car B. h indicates the separation between the two cars. Initially $x_{AF} = 0$ and $x_{BR} = 9\,\mathrm{m}$

We'll break the analysis into two phases. First we'll determine the position and speed of the vehicles at $t = 1\,\mathrm{s}$. Next, we'll "reset" our time axis, defining $t = 0$ as the point at which vehicle A begins to decelerate. We'll then find the time to contact and the relative speed at contact.

At $t = 0$ we have

$$\begin{array}{ll} x_{AF} = 0, & x_{BR} = 9\,\mathrm{m} \\ v_A = 30\,\mathrm{m/s} & v_B = 30\,\mathrm{m/s} \\ a_A = 0, & a_B = -10.7\,\mathrm{m/s^2} \end{array}$$

At $t = 1\,\mathrm{s}$ we have

$$\begin{array}{ll} x_{AF} = (30\,\mathrm{m/s})(1\,\mathrm{s}) = 30\,\mathrm{m}, & x_{BR} = 9\,\mathrm{m} + (30\,\mathrm{m/s})(1\,\mathrm{s}) - \frac{10.7\,\mathrm{m/s^2}(1\,\mathrm{s})^2}{2} = 33.65\,\mathrm{m} \\ v_A = 30\,\mathrm{m/s} & v_B = (30\,\mathrm{m/s}) - (10.7\,\mathrm{m/s^2})(1\,\mathrm{s}) = 19.3\,\mathrm{m/s} \end{array}$$

Note that by the time the driver of Car A notices the problem ahead of him and initiates his braking (just 1 s delay), his distance from the car ahead of him has been reduced from 9 m to $(33.65 - 30)\,\mathrm{m} = 3.65\,\mathrm{m}$.

We'll now "reset" our system, starting t from zero again but with the new values of position and speed, as well as having both cars decelerate (Car A at $9\,\mathrm{m/s^2}$ and Car B continuing at $10.7\,\mathrm{m/s^2}$).

$$x_{AF}(t) = 30\,\mathrm{m} + (30\,\mathrm{m/s})t - \frac{(9\,\mathrm{m/s^2})t^2}{2}, \quad x_{BR}(t) = 33.65\,\mathrm{m} + (19.3\,\mathrm{m/s})t - \frac{(10.7\,\mathrm{m/s^2})t^2}{2} \tag{1}$$

$$v_A(t) = 30\,\mathrm{m/s} - (9\,\mathrm{m/s^2})t \qquad v_B(t) = 19.3\,\mathrm{m/s} - (10.7\,\mathrm{m/s^2})t \tag{2}$$

Solving $x_{BR} - x_{AF} = 0$ will allow us to see whether a collision (separation of the two cars goes to zero) occurs.

$$3.65\,\mathrm{m} - (10.7\,\mathrm{m/s})t^* - \frac{(1.7\,\mathrm{m/s^2})t^{*2}}{2} = 0$$

This has only one positive solution: $\boxed{t^* = 0.332\,\mathrm{s}}$. Thus we do have a collision.

Using this value of time and (2) gives us a collision speed of

$$v_{coll} = v_A - v_B = 10.7\,\mathrm{m/s} + (1.7\,\mathrm{m/s^2})(0.332\,\mathrm{s}) = 11.26\,\mathrm{m/s}$$

$$\boxed{v_{coll} = 11.26\,\mathrm{m/s} = 40.54\,\mathrm{km/hr}}$$

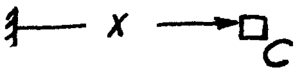
Note that these figures are not unreasonable. People often travel far too close behind the car ahead of them and this shows that even a very brief lack of attention can easily lead to a collision.

2.1.24

GOAL: Determine the performance difference between manual braking and using Brake Assist.

GIVEN: In each case the car decelerates from 96 km/h. Without Brake Assist the car decelerates at $6\,\mathrm{m/s^2}$ for $1\,\mathrm{s}$ and then at $9\,\mathrm{m/s^2}$ for the remaining time. With Brake Assist active the car decelerates at $9\,\mathrm{m/s^2}$ the entire time. Let x represent the position of the car C.

DRAW:



FORMULATE EQUATIONS:

We'll use the formulas for motion with constant acceleration a (initial position and speed given by x_0, v_0):

$$v(t) = v_0 + at$$
$$x(t) = x_0 + v_0 t + \frac{at^2}{2}$$

SOLVE:

Case 1: Without Brake Assist. For this case we'll examine the car's response in two phases - from $t = 0\,\mathrm{s}$ to $t = 1\,\mathrm{s}$ and then for $t > 1\,\mathrm{s}$. We'll "reset" our time axis for the second phase, remembering that $1\,\mathrm{s}$ has already elapsed.

At $t = 0$ we have

$$\begin{aligned} x =& \;\; 0\,\text{meter} \\ v =& \;\; 96\,\mathrm{km/h} = \tfrac{96\times10^3}{3600}\,\mathrm{m/s} = 26.7\,\mathrm{m/s} \\ a =& \;\; -6\,\mathrm{m/s^2} \end{aligned}$$

At $t = 1\,\mathrm{s}$ we have

$$\begin{aligned} x &= (26.7\,\mathrm{m/s})(1\,\mathrm{s}) - 0.5(6\,\mathrm{m/s^2})(1\,\mathrm{s})^2 = 23.7\,\mathrm{m} \\ v &= (26.7\,\mathrm{m/s}) - (6\,\mathrm{m/s^2})(1\,\mathrm{s}) = 20.7\,\mathrm{m/s} \end{aligned}$$

We'll now "reset" our system, starting t from zero again but with the new values of position and speed as initial conditions.

$$x(t) = 23.7\,\mathrm{m} + (20.7\,\mathrm{m/s})t - \frac{(9\,\mathrm{m/s^2})t^2}{2} \tag{1}$$
$$v(t) = 20.7\,\mathrm{m/s} - (9\,\mathrm{m/s^2})t \tag{2}$$

Solving $v(t^*) = 0$ will tell us when the car's speed goes to zero:

(2) $\Rightarrow$
$$(9\,\mathrm{m/s^2})t^* = 20.7\,\mathrm{m/s} \Rightarrow t^* = 2.3\,\mathrm{s}$$

(1) $\Rightarrow$
$$x(2.3\,\mathrm{s}) = 23.7\,\mathrm{m} + (20.7\,\mathrm{m/s})(2.3\,\mathrm{s}) - \frac{(9\,\mathrm{m/s^2})(2.3\,\mathrm{s})^2}{2} = 47.6\,\mathrm{m}$$

Case 2: With Brake Assist we have

$$x(t) = (26.7\,\mathrm{m/s})t - \frac{(9\,\mathrm{m/s^2})t^2}{2} \tag{3}$$
$$v(t) = 26.7\,\mathrm{m/s} - (9\,\mathrm{m/s^2})t \tag{4}$$

Solving for $v(t^*) = 0$ gives us

(4) $\Rightarrow$
$$(9\,\mathrm{m/s^2})t^* = 26.7\,\mathrm{m/s} \Rightarrow t^* = 2.97\,\mathrm{s}$$

(3) $\Rightarrow$
$$x(2.97\,\mathrm{s}) = (26.7\,\mathrm{m/s})(2.97\,\mathrm{s}) - \frac{(9\,\mathrm{m/s^2})(2.97\,\mathrm{s})^2}{2} = 39.6\,\mathrm{m}$$

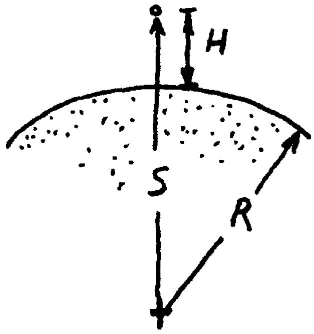
What we see from comparing these two results is that without Brake Assist the car traveled $\boxed{8\,\mathrm{m}}$ farther, an increase of $\boxed{17\%}$.

2.1.25

GOAL: Determine the impact speed.

GIVEN: The battlestation falls from rest at a height $H = 644\,\text{km}$. The planet has radius $R = 6440\,\text{km}$ and a gravitational acceleration of $g_0 = 9\,\text{m/s}^2$.

DRAW:



FORMULATE EQUATIONS:

The battlestation falls according to

$$\ddot{s} = -\frac{g_0 R^2}{s^2} \tag{1}$$

The acceleration is given in terms of the position s, so

$$\begin{aligned} \ddot{s} = \frac{d\dot{s}}{dt} = \frac{d\dot{s}}{ds}\cdot\frac{ds}{dt} = \frac{d\dot{s}}{ds}\dot{s} \\ \dot{s}d\dot{s} = \ddot{s}ds \end{aligned} \tag{2}$$

SOLVE:

(1) $\rightarrow$ (2), integrate $\Rightarrow$

$$\int_0^{\dot{s}_f} \dot{s}d\dot{s} = -\int_{H+R}^{R} \frac{g_0 R^2}{s^2} ds$$

$$\begin{aligned} \frac{1}{2}\dot{s}^2\Big|_0^{\dot{s}_f} &= -g_0 R^2 \int_{H+R}^{R} \frac{ds}{s^2} \\ \frac{1}{2}(\dot{s}_f)^2 &= \frac{g_0 R^2}{s}\Big|_{H+R}^{R} \\ \dot{s}_f &= \sqrt{2 g_0 R^2 \left[\frac{1}{R} - \frac{1}{H+R}\right]} \end{aligned}$$

$$\dot{s}_f = \sqrt{2(9\,\mathrm{m/s^2})(6440 \times 10^3\,\mathrm{m})^2 \left[\frac{1}{6440 \times 10^3\,\mathrm{m}} - \frac{1}{(6440 + 644) \times 10^3\,\mathrm{m}}\right]}$$

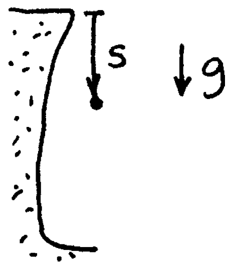
$$\boxed{|\dot{s}_f| = 3246.26\,\mathrm{m/s} = 11685.6\,\mathrm{km/hr}}$$

2.1.27

GOAL: Find a rock's impact velocity when dropped from 30 meter.

GIVEN: Downward acceleration is $g - c_d\dot{s}^2$

DRAW:



ASSUME: The rock experiences accelerations due only to that of gravity and drag.

FORMULATE EQUATIONS:

Measuring from the clifftop we have

$$s(0) = 0 \text{ and } \dot{s}(0) = 0$$

Because of the air resistance our ball's acceleration is

$$\ddot{s}(t) = 9.81\,\mathrm{m/s^2} - (0.01\,\mathrm{m^{-1}})\dot{s}^2$$

Using $\ddot{s}ds = \dot{s}d\dot{s}$ or

$$ds = \frac{\dot{s}d\dot{s}}{9.81\,\mathrm{m/s^2} - (0.01\,\mathrm{m^{-1}})\dot{s}^2}$$

SOLVE:

Integrating gives

$$\int_0^{30\text{ m}} ds = \int_0^{\dot{s}_{impact}} \frac{\dot{s}d\dot{s}}{9.81\,\mathrm{m/s^2} - (0.01\,\mathrm{m^{-1}})\dot{s}^2}$$

$$30\,\mathrm{m} = -\frac{1\,\mathrm{m}}{0.02}\ln(9.8\,\mathrm{m/s^2} - 0.01\,\mathrm{m^{-1}})\dot{s}^2|_0^{\dot{s}_{impact}}$$

$$-0.6 = \ln\left[\frac{9.8\,\mathrm{m/s^2} - (0.01\,\mathrm{m^{-1}})\dot{s}_{impact}^2}{9.8\,\mathrm{m/s^2}}\right]$$

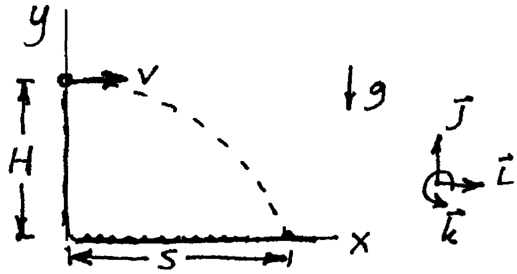
$$\boxed{\dot{s}_{impact} = \sqrt{(1 - e^{0.6})\frac{9.8}{0.01}} = 7.555\,\mathrm{m/s}}$$

2.2.4

GOAL: Determine the horizontal location before the island at which to drop the supplies.

GIVEN: System geometry and initial speed: $H = 46\,\mathrm{m}$, $v = 322\,\mathrm{km/hr}$.

DRAW:



FORMULATE EQUATIONS:

The x position of the airlifted supplies is given by

$$x = vt \tag{1}$$

The y position of the supplies is given by

$$y = H - \frac{1}{2}gt^2 \tag{2}$$

SOLVE:

The total horizontal distance traveled is s, so (1) becomes

(1) $\Rightarrow$
$$s = vt \tag{3}$$

To find the time when the supplies land, set $y = 0$ in (2), so

(2) $\Rightarrow$
$$0 = H - \frac{1}{2}gt^2 \tag{4}$$

(4) $\Rightarrow$
$$t = \sqrt{\frac{2H}{g}} \tag{5}$$

Eliminate time in (3) to find the distance:

(5) $\rightarrow$ (3) $\Rightarrow$
$$s = v\sqrt{\frac{2H}{g}}$$

$$s = (322\,\mathrm{km/hr})\left(\frac{\mathrm{hr}}{3600\,\mathrm{s}}\right)\left(\frac{10^3\,\mathrm{m}}{\mathrm{km}}\right)\sqrt{\frac{2(46\,\mathrm{m})}{9.81\,\mathrm{m/s^2}}}$$

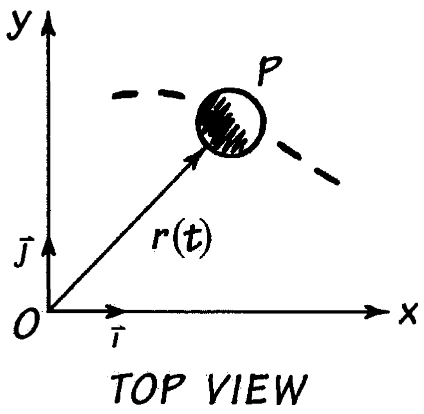
$$\boxed{s = 273.9\,\mathrm{m}}$$

2.2.5

GOAL: Determine $\|\vec{r}(t)\|$, $\|\vec{v}(t)\|$, and $\|\vec{a}(t)\|$ at time $t = 1\,\text{s}$.

GIVEN: The position of particle P in the horizontal plane is described by $\vec{r}(t) = 0.09t^3\vec{i} + 0.12t^2 \sin 2t\vec{j}$.

DRAW:



FORMULATE EQUATIONS:

The velocity and acceleration vectors for particle P are defined as, respectively,

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \tag{1}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} \tag{2}$$

SOLVE:

The particle's distance from the origin at $t = 1\,\text{s}$ is

$$\|\vec{r}(t)\| = \sqrt{\vec{r}(t) \cdot \vec{r}(t)} = \sqrt{(0.09t^3)^2 + (0.12t^2 \sin 2t)^2}\,\text{m}$$

$$\|\vec{r}(1)\| = \sqrt{[0.09(1)^3]^2 + [0.12(1)^2 \sin(114.59 \times 1)]^2}\,\text{m}$$

$$\boxed{\|\vec{r}(1)\| = 0.1414\,\text{m}}$$

The speed of the particle at the given instant is

(1) $\Rightarrow$

$$\vec{v}(t) = 0.27t^2\vec{i} + 0.24t \sin 2t\vec{j} + 0.24t^2 \cos 2t\vec{j}\ \text{m/s}$$

$$\vec{v}(t) = 0.27t^2\vec{i} + \left(0.24t\sin 2t + 0.24t^2\cos 2t\right)\vec{j} \ \text{m/s}$$

$$\|\vec{v}(t)\| = \sqrt{\vec{v}(t)\cdot\vec{v}(t)} = \sqrt{(0.27t^2)^2 + (0.24t\sin 2t + 0.24t^2\cos 2t)^2}\ \text{m/s}$$

$$\|\vec{v}(1)\| = \sqrt{[(0.27(1)^2)]^2 + [0.24(1)\sin(114.59\times 1) + 0.24(1)^2\cos(114.59\times 1)]^2}\ \text{m/s}$$

$$\boxed{\|\vec{v}(1)\| = 0.295\,\text{m/s}}$$

The particle's acceleration at $t = 1\,\text{s}$ is given by

$$(2) \Rightarrow \quad \vec{a}(t) = 0.54t\vec{i} + 0.48t\cos 2t\vec{j} + 0.48t\cos 2t\vec{j} + 0.24\sin 2t\vec{j} - 0.48t^2\sin 2t\vec{j} \ \text{m/s}^2$$

$$\vec{a}(t) = 0.54t\vec{i} + \left(0.24\sin 2t + 0.96t\cos 2t - 0.48t^2\sin 2t\right)\vec{j} \ \text{m/s}^2$$

$$\|\vec{a}(t)\| = \sqrt{\vec{a}(t)\cdot\vec{a}(t)} = \sqrt{(0.54t)^2 + (0.24\sin 2t + 0.96t\cos 2t - 0.48t^2\sin 2t)^2}\ \text{m/s}^2$$

$$\|\vec{a}(1)\| = \sqrt{[0.54(1)]^2 + [0.24\sin(114.59\times 1) + 0.96(1)\cos(114.59\times 1) - 0.48(1)^2\sin(114.59\times 1)]^2}\ \text{m/s}^2$$

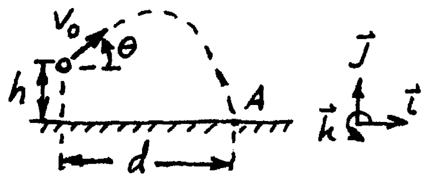
$$\boxed{\|\vec{a}(1)\| = 0.82\,\text{m/s}^2}$$

2.2.9

GOAL: Determine the minimum launch angle and corresponding time of flight.

GIVEN: System geometry and initial speed: $d = 60\,\text{m}$, $h = 0.9\,\text{m}$, $v_0 = 30\,\text{m/s}$.

DRAW:



FORMULATE EQUATIONS:

The x position of the pumpkin is given by

$$x = (v_0 \cos\theta)t \tag{1}$$

The y position of the pumpkin is given by

$$y = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

SOLVE:

The total horizontal distance traveled is d, so (1) becomes

(1) $\Rightarrow$

$$d = (v_0 \cos\theta)t \tag{3}$$

(3) $\Rightarrow$

$$t = \frac{d}{(v_0 \cos\theta)} \tag{4}$$

To find the launch angle, first set $y = 0$ in (2), so

(2) $\Rightarrow$

$$0 = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{5}$$

Then eliminate the time t:

(4) $\rightarrow$ (5) $\Rightarrow$

$$0 = h + \frac{\sin\theta}{\cos\theta}d - \frac{1}{2}g\left(\frac{d}{v_0 \cos\theta}\right)^2$$

$$
\begin{aligned}
0 &= h\cos^2\theta + d\sin\theta\cos\theta - \frac{gd^2}{2v_0^2} \\
0 &= h\cos^2\theta + \frac{1}{2}d\sin(2\theta) - \frac{gd^2}{2v_0^2} \\
0 &= 2h\cos^2\theta + d\sin(2\theta) - \frac{gd^2}{v_0^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{gd^2}{v_0^2} &= 2h\cos^2\theta + d\sin(2\theta) \\
\frac{(9.81\,\mathrm{m/s^2})(60\,\mathrm{m})^2}{(30\,\mathrm{m/s})^2} &= 2(0.9\,\mathrm{m})\cos^2\theta + (60\,\mathrm{m})\sin(2\theta) \\
19.62\,\mathrm{m} &= 1.8\cos^2\theta\,\mathrm{m} + 60\sin(2\theta)\,\mathrm{m} \qquad (6)
\end{aligned}
$$

Using MATLAB® to solve for the minimum angle that satisfies (6),

$$\boxed{\theta_{\mathrm{min}} = 0.333\,\mathrm{rad} = 19.06^\circ} \qquad (7)$$

(7) → (4) ⇒

$$t = \frac{60\,\mathrm{m}}{(30\,\mathrm{m/s})(\cos(19.06^\circ))}$$

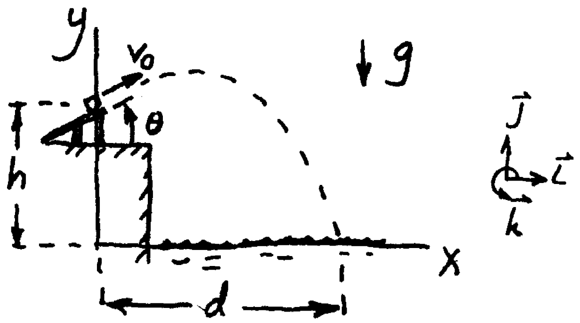
$$\boxed{t = 2.116\,\mathrm{s}}$$

2.2.10

GOAL: Determine the time of flight, the horizontal distance traveled, and the maximum height achieved.

GIVEN: System geometry and initial speed: $h = 6\,\mathrm{m}$, $\theta = 45°$, $v_0 = 7.6\,\mathrm{m/s}$.

DRAW:



FORMULATE EQUATIONS:

The x position of the craft is given by

$$x = (v_0 \cos\theta)t \tag{1}$$

The y position of the craft is given by

$$y = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

SOLVE:

Find the time of flight by setting $y = 0$ in (2) and solving for t:

(2) $\Rightarrow$
$$0 = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{3}$$

(3) $\Rightarrow$
$$t = \frac{-v_0 \sin\theta \pm \sqrt{(v_0 \sin\theta)^2 + 2gh}}{-g}$$

$$\begin{aligned} t &= \frac{-(7.6\,\mathrm{m/s})\sin(45°) \pm \sqrt{((7.6\,\mathrm{m/s})\sin(45°))^2 + 2(9.81\,\mathrm{m/s^2})(6\,\mathrm{m})}}{-9.81\,\mathrm{m/s^2}} \\ t &= -0.686\,\mathrm{s},\ 1.782\,\mathrm{s} \end{aligned}$$

$$\boxed{t = 1.782\,\mathrm{s}} \tag{4}$$

Use the time of flight to solve for the distance traveled:

(4) $\rightarrow$ (1) $\Rightarrow$

$$x = (7.6\,\mathrm{m/s})\cos(45°)(1.782\,\mathrm{s})$$

$$\boxed{x = 9.58\,\mathrm{m}}$$

Determine the maximum height achieved by first differentiating (2) and setting it equal to 0:

Differentiate (2) $= 0 \Rightarrow$

$$\frac{dy}{dt} = v_0 \sin\theta - gt = 0 \tag{5}$$

(5) $\Rightarrow$

$$t_{\text{peak}} = \frac{v_0 \sin\theta}{g} \tag{6}$$

(6) $\rightarrow$ (2) $\Rightarrow$

$$y_{\max} = h + \frac{(v_0 \sin\theta)^2}{g} - \frac{(v_0 \sin\theta)^2}{2g}$$

$$\begin{aligned} y_{\max} &= h + \frac{(v_0 \sin\theta)^2}{2g} \\ y_{\max} &= 6\,\mathrm{m} + \frac{((7.6\,\mathrm{m/s})\sin(45°))^2}{2(9.81\,\mathrm{m/s^2})} \end{aligned}$$

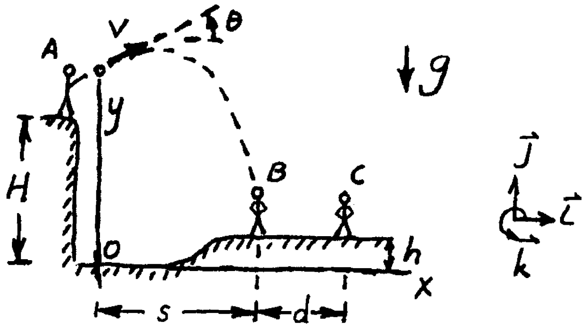
$$\boxed{y_{\max} = 7.47\,\mathrm{m}}$$

2.2.11

GOAL: Determine the separation distance and the constant running speed needed to catch the ball.

GIVEN: System geometry and initial speed: $H = 9\,\mathrm{m}$, $h = 3\,\mathrm{m}$, $d = 6\,\mathrm{m}$, $\theta = 30°$, $v = 6\,\mathrm{m/s}$.

DRAW:



FORMULATE EQUATIONS:

The x position of the ball is given by

$$x = (v\cos\theta)t \tag{1}$$

The y position of the ball is given by

$$y = H + (v\sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

The x position of the catcher is given by

$$x^* = x_0^* + v^* t \tag{3}$$

SOLVE:

The separation distance is s, so (1) becomes

(1) $\Rightarrow$

$$s = (v\cos\theta)t \tag{4}$$

We need the time of flight to solve for s, so set $y = h$ in (2) and solve for t:

(2) $\Rightarrow$

$$h = H + (v\sin\theta)t - \frac{1}{2}gt^2$$
$$0 = -\frac{1}{2}gt^2 + (v\sin\theta)t + (H - h) \tag{5}$$

(5) $\Rightarrow$

$$t = \frac{-v\sin\theta \pm \sqrt{(v\sin\theta)^2 + 2g(H-h)}}{-g}$$

$$\begin{aligned} t &= \frac{-(6\,\mathrm{m/s})\sin(30^\circ) \pm \sqrt{((6\,\mathrm{m/s})\sin(30^\circ))^2 + 2(9.81\,\mathrm{m/s^2})(9\,\mathrm{m} - 3\,\mathrm{m})}}{-9.81\,\mathrm{m/s^2}} \\ t &= -0.842\,\mathrm{s},\ 1.453\,\mathrm{s} \end{aligned}$$

$$\boxed{t = 1.453\,\mathrm{s}} \tag{6}$$

(6) $\rightarrow$ (4) $\Rightarrow$

$$s = (6\,\mathrm{m/s})\cos(30^\circ)(1.453\,\mathrm{s})$$

$$\boxed{s = 7.55\,\mathrm{m}}$$

To find the running speed,

(3) $\Rightarrow$

$$x^* - x_0^* = v^* t$$

$$\begin{aligned} -d &= v^* t \\ v^* &= -\frac{d}{t} \\ v^* &= -\frac{6\,\mathrm{m}}{1.453\,\mathrm{s}} \end{aligned}$$

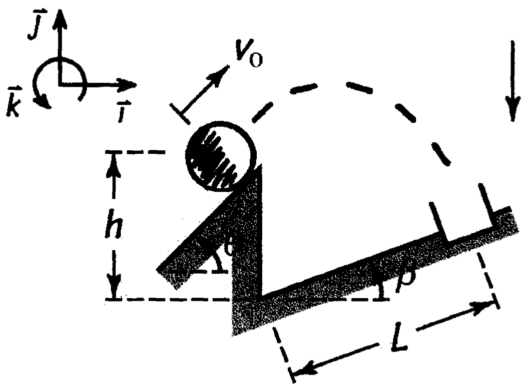
$$\boxed{v^* = -4.13\,\mathrm{m/s}\ \text{(running to the left)}}$$

2.2.12

GOAL: Determine the ball's launch speed and how long it is airborne.

GIVEN: The launch ramp is angled at $\theta = 45°$ with respect to the ground, and its end is located $h = 0.6\,\text{m}$ above the base of the backboard, which is inclined at $\beta = 20°$ to the horizontal. The hole is $L = 0.9\,\text{m}$ along the backboard from its base.

DRAW:



FORMULATE EQUATIONS:

The x position of the ball is given by

$$x = x_0 + \dot{x}_0 t \tag{1}$$

The y position of the ball is given by

$$y = y_0 + \dot{y}_0 t - \frac{1}{2} g t^2 \tag{2}$$

SOLVE:

We'll need to eliminate the time t to first find the launch speed v_0, and so from (1) we get that

(1) $\Rightarrow$

$$L \cos\beta = (v_0 \cos\theta)\, t$$

$$t = \frac{L\cos\beta}{v_0\cos\theta} \tag{3}$$

By eliminating the time t in (2), we find that the ball's launch speed v_0 must be

(3) $\rightarrow$ (2) $\Rightarrow$

$$L\sin\beta = h + (v_0\sin\theta)\left(\frac{L\cos\beta}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{L\cos\beta}{v_0\cos\theta}\right)^2$$

$$(L\sin\beta - h)\cos^2\theta = L\cos\beta\,(\sin\theta\cos\theta) - \frac{gL^2\cos^2\beta}{2v_0^2}$$

$$\frac{gL^2\cos^2\beta}{2v_0^2} = (h - L\sin\beta)\cos^2\theta + \frac{1}{2}L\cos\beta\sin 2\theta$$

$$v_0 = \sqrt{\frac{gL^2\cos^2\beta}{2\,(h - L\sin\beta)\cos^2\theta + L\cos\beta\sin 2\theta}}$$

$$v_0 = \sqrt{\frac{(9.81\,\mathrm{m/s^2})(0.9\,\mathrm{m})^2\cos^2(20^\circ)}{2\,[0.6\,\mathrm{m} - (0.9\,\mathrm{m})\sin(20^\circ)]\cos^2(45^\circ) + (0.9\,\mathrm{m})\cos(20^\circ)\sin(2\times 45^\circ)}}$$

$$\boxed{v_0 = 2.48\,\mathrm{m/s}}$$

Therefore, the ball's time of flight is

(3) $\Rightarrow$

$$t = \frac{(0.9\,\mathrm{m})\cos(20^\circ)}{(2.48\,\mathrm{m/s})\cos(45^\circ)}$$

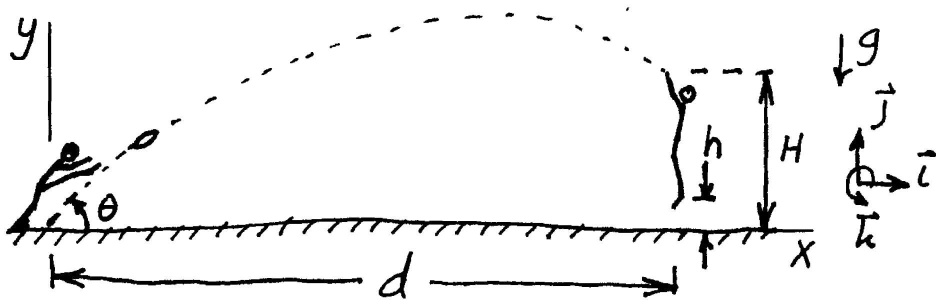
$$\boxed{t = 0.482\,\mathrm{s}}$$

2.2.13

GOAL: Determine when the football player should jump to catch the ball and how far away he is.

GIVEN: System geometry, the football's initial speed, and the catcher's jump speed: $H = 2.1\,\text{m}$, $h = 0.3\,\text{m}$, $\theta = 50°$, $v_0 = 12\,\text{m/s}$, $v_j = 2.7\,\text{m/s}$.

DRAW:



FORMULATE EQUATIONS:

The x position of the football is given by

$$x = (v_0 \cos\theta)t \tag{1}$$

The y position of the football is given by

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

The y position of the catcher is given by

$$y_j = y_0 + v_j t_j - \frac{1}{2}gt_j^2 \tag{3}$$

SOLVE:

First find the total time taken for the football to reach the catching height H by setting $y = H$ in (2) and solving for t:

(2) $\Rightarrow$

$$H = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$0 = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t - H \tag{4}$$

(4) $\Rightarrow$

$$t = \frac{-v_0 \sin\theta \pm \sqrt{(v_0 \sin\theta)^2 - 2gH}}{-g}$$

$$\begin{aligned}
t &= \frac{-(12\,\mathrm{m/s})\sin(50^\circ) \pm \sqrt{((12\,\mathrm{m/s})\sin(50^\circ))^2 - 2(9.81\,\mathrm{m/s^2})(2.1\,\mathrm{m})}}{-9.81\,\mathrm{m/s^2}} \\
t &= 0.266\,\mathrm{s},\ 1.608\,\mathrm{s}
\end{aligned}$$

$$t = 1.608\,\mathrm{s} \text{ when the ball reaches } H \text{ on the way down}$$

Next find the time it takes the player to jump and catch the football:

(3) $\Rightarrow$

$$y_j - y_0 = v_j t_j - \frac{1}{2} g t_j^2$$

$$\begin{aligned}
h &= v_j t_j - \frac{1}{2} g t_j^2 \\
0 &= -\frac{1}{2} g t_j^2 + v_j t_j - h \qquad (5)
\end{aligned}$$

(5) $\Rightarrow$

$$t_j = \frac{-v_j \pm \sqrt{v_j^2 - 2gh}}{-g}$$

$$\begin{aligned}
t_j &= \frac{-2.7\,\mathrm{m/s} \pm \sqrt{(2.7\,\mathrm{m/s})^2 - 2(9.81\,\mathrm{m/s^2})(0.3\,\mathrm{m})}}{-9.81\,\mathrm{m/s^2}} \\
t_j &= 0.154\,\mathrm{s},\ 0.396\,\mathrm{s}
\end{aligned}$$

$$t_j = 0.154\,\mathrm{s} \text{ when the catcher reaches } H \text{ on the way up}$$

The catcher needs to jump at

$$t^* = t - t_j = 1.608\,\mathrm{s} - 0.154\,\mathrm{s}$$

$$\boxed{t^* = 1.454\,\mathrm{s}}$$

The distance the ball travels before being caught is d, so

(1) $\Rightarrow$

$$d = (v_0 \cos\theta)t$$

$$d = (12\,\mathrm{m/s})(\cos(50^\circ))(1.608\,\mathrm{s})$$

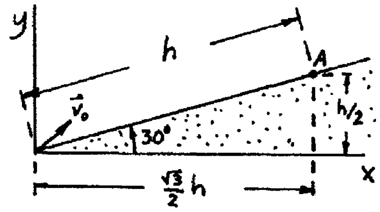
$$\boxed{d = 12.4\,\mathrm{m}}$$

2.2.18

GOAL: Find where a cricket lands on an inclined slope

GIVEN: Angle of slope.

DRAW:



FORMULATE EQUATIONS:

$$x = v_0 t \cos\eta$$

$$y = v_0 t \sin\eta - \frac{1}{2}gt^2$$

$$\text{at } A: x = \frac{\sqrt{3}}{2}h, y = \frac{h}{2}$$

SOLVE:

$$\frac{\sqrt{3}}{2}h = v_0 t^* \cos\eta \quad (1)$$

$$\frac{h}{2} = v_0 t^* \sin\eta - \frac{1}{2}g(t^*)^2 \quad (2)$$

(1) $\Rightarrow$

$$t = \frac{\sqrt{3}h}{2v_0\cos\eta} \quad (3)$$

(3) $\rightarrow$ (2) $\Rightarrow$

$$\frac{h}{2} = \frac{\sqrt{3}h\sin\eta}{2\cos\eta} - \frac{1}{2}g\frac{3h^2}{4v_0^2\cos^2\eta}$$

$$\sqrt{3}\sin\eta\cos\eta - (\cos\eta)^2 - \frac{3}{4}\frac{hg}{v_0^2} = 0$$

Let $f(\eta) = \sqrt{3}\sin\eta\cos\eta - \cos^2\eta - \frac{3}{4}\frac{hg}{v_0^2}$ and find the value(s) of η for which $f(\eta) = 0$

Using the Newton-Raphson procedure to solve for the roots gives:

$$\frac{df(\eta)}{d\eta} = \sqrt{3}\cos(2\eta) + 2\cos\eta\sin\eta = \sqrt{3}\cos(2\eta) + \sin(2\eta)$$

$$\eta_{i+1} = \eta_i - \frac{f}{\frac{df}{d\eta}} = \eta_i - \frac{\frac{\sqrt{3}}{2}\sin(2\eta) - \cos^2\eta - 0.3704}{\sqrt{3}\cos(2\eta) + \sin(2\eta)} \tag{4}$$

Initial guess: $\eta_0 = 0.7$. Using (4) yields $\eta_1 = 0.7798$. Using (4) again yields $\eta_2 = 0.7893$. One last iteration gives $\eta_3 = 0.7902$.

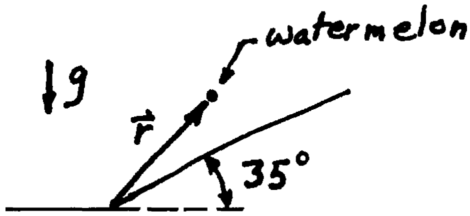
$$\boxed{\eta = 0.7902 \text{ rad} = 45.3°}$$

2.2.19

GOAL: Determine initial launch speed for a launched watermelon to impact a slope at a specified position.

GIVEN: Angle of slope and angle of launch.

DRAW:



FORMULATE EQUATIONS:

$$\vec{r}(t) = \left(-\frac{1}{2}gt^2\right)\vec{j} + \vec{v}(0)t + \vec{r}(0) \tag{1}$$

ASSUME:

$$\begin{aligned} \vec{r}(t_f) &= 3(\cos(35°)\vec{i} + \sin(35°)\vec{j})\,\mathrm{m} \\ \vec{v}(0) &= v_0(\cos(45°)\vec{i} + \sin(45°)\vec{j})\,\mathrm{m/s} \end{aligned} \tag{2}$$

SOLVE:

(2)→(1)⇒

$$3(\cos(35°)\vec{i} + \sin(35°)\vec{j})\,\mathrm{m} = \left(-\frac{1}{2}gt_f^2\right)\vec{j} + v_0(\cos(45°)\vec{i} + \sin(45°)\vec{j})t_f$$

$\vec{i}$:

$$\begin{aligned} 3\cos(35°)\,\mathrm{m} &= v_0\cos(45°)t_f \\ t_f &= \frac{3\cos(35°)}{v_0\cos(45°)}\,\mathrm{sec} \end{aligned} \tag{3}$$

$\vec{j}$:

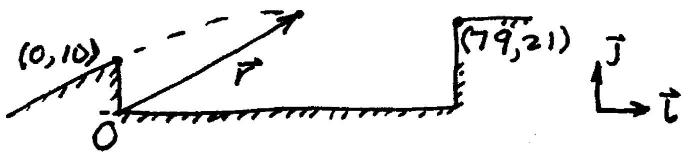
$$\begin{aligned} 3\sin(35°)\,\mathrm{m} &= v_0\sin(45°)t_f - \frac{1}{2}gt_f^2 \\ 3\sin(35°)\,\mathrm{m} &= v_0\sin(45°)\left(\frac{3\cos(35°)}{v_0\cos(45°)}\,\mathrm{sec}\right) - \frac{1}{2}g\left(\frac{3\cos(35°)}{v_0\cos(45°)}\,\mathrm{sec}\right)^2 \\ v_0 &= \sqrt{\frac{(-\frac{1}{2})(9.81\,\mathrm{m/s^2})\left(\frac{3\cos(35°)}{\cos(45°)}\,\mathrm{m}\right)^2}{3(\sin(35°) - \tan(45°)\cos(35°))\,\mathrm{m}}} \\ v_0 &= \boxed{8.97\ \mathrm{m/s}} \end{aligned}$$

2.2.22

GOAL: Calculate minimum speed v to complete stunt.

GIVEN: Positions at which the car leaves the ramp, angle of the ramp and target location.

DRAW:



FORMULATE EQUATIONS:

$$\vec{r}(t) = \frac{1}{2}t^2\vec{a} + \vec{v}(0)t + \vec{r}(0) \tag{1}$$

ASSUME:

$$\begin{aligned} \vec{v}(0) &= v(\cos(20°)\vec{i} + \sin(20°)\vec{j} \\ \vec{r}(0) &= 3\vec{j} \text{ m} \\ \vec{r}(t_f) &= (24\vec{i} + 6.4\vec{j})\,\text{m} \end{aligned} \tag{2}$$

SOLVE:

(2)→(1)⇒ $$-g\frac{1}{2}t^2\vec{j} + v(\cos(20°)\vec{i} + \sin(20°)\vec{j})t + 3\vec{j} \text{ m} = 24\vec{i} \text{ m} + 6.4\vec{j} \text{ m}$$

$\vec{i}$: $$\begin{aligned} v\cos(20°)t &= 24\,\text{m} \\ t &= \frac{24\,\text{m}}{v\cos(20°)} \end{aligned} \tag{3}$$

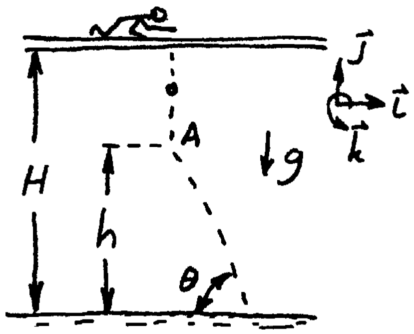
$\vec{j}$: $$\begin{aligned} 21 \text{ ft} &= -\frac{g}{2}\left(\frac{24\,\text{m}}{v\cos(20°)}\right)^2 + v\sin(20°)\left(\frac{24\,\text{m}}{v\cos(20°)}\right) + 3 \text{ m} \\ v &= \frac{24\,\text{m}}{\cos(20°)}\sqrt{\frac{9.81\,\text{m/s}^2}{2((24 \text{ m})\tan(20°) - 3.4\,\text{m})}} \\ v &= 24.5\,\text{m/s} = \boxed{88.2 \text{ km/h}} \end{aligned}$$

2.2.23

GOAL: Determine the time it takes the ball to hit the water's surface, where it hits, the impact speed, and the angle of impact.

GIVEN: The ball starts from rest at $H = 21\,\mathrm{m}$. The wind speed is $v_w = 24\,\mathrm{m/s}$ at $h = 15\,\mathrm{m}$.

DRAW:



FORMULATE EQUATIONS:

The y position of the ball is given by

$$y = H - \frac{1}{2}gt^2 \tag{1}$$

The time it takes for the ball to fall from where the wind pushes it to the water's surface is

$$t^* = t - t_A \tag{2}$$

The horizontal distance traveled due to the wind is

$$s = v_w t^* \tag{3}$$

Since it starts from rest, the ball's final vertical speed is given by

$$\dot{y}_f^2 = 2gH \tag{4}$$

SOLVE:

Find the time to impact by setting $y = 0$ in (1):

(1) ⇒
$$0 = H - \frac{1}{2}gt^2$$

$$
\begin{aligned}
t &= \sqrt{\frac{2H}{g}} \\
t &= \sqrt{\frac{2(21\,\mathrm{m})}{9.81\,\mathrm{m/s^2}}}
\end{aligned}
$$

$$\boxed{t = 2.069\,\mathrm{s}}$$

The time it takes to fall to where the wind pushes the ball (point A) is

(1) $\Rightarrow$

$$h = H - \frac{1}{2}gt_A^2$$

$$
\begin{aligned}
t_A &= \sqrt{\frac{2(H-h)}{g}} \\
t_A &= \sqrt{\frac{2(21\,\mathrm{m} - 15\,\mathrm{m})}{9.81\,\mathrm{m/s^2}}} \\
t_A &= 1.106\,\mathrm{s}
\end{aligned}
$$

From point A to the water's surface,

(2) $\Rightarrow$

$$t^* = 2.069\,\mathrm{s} - 1.106\,\mathrm{s}$$

$$t^* = 0.963\,\mathrm{s}$$

The distance traveled due to the wind is then

(3) $\Rightarrow$

$$s = (24\,\mathrm{m/s})(0.963\,\mathrm{s})$$

$$\boxed{s = 23.1\,\mathrm{m}}$$

The impact speed is

$$v_f = \sqrt{\dot{x}_f^2 + \dot{y}_f^2} \tag{5}$$

The final horizontal speed is just the wind speed, so

(4) $\rightarrow$ (5) $\Rightarrow$

$$v_f = \sqrt{v_w^2 + 2gH}$$

$$v_f = \sqrt{(24\,\mathrm{m/s})^2 + 2(9.81\,\mathrm{m/s^2})(21\,\mathrm{m})}$$

$$\boxed{|v_f| = 31.43\,\mathrm{m/s}}$$

The impact angle is

$$\theta = \tan^{-1}\left(\frac{\dot{y}_f}{\dot{x}_f}\right) \tag{6}$$

$(4) \rightarrow (6) \Rightarrow$

$$\theta = \tan^{-1}\left(\frac{\sqrt{2gH}}{v_w}\right)$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{2(9.81\,\mathrm{m/s^2})(21\,\mathrm{m})}}{24\,\mathrm{m/s}}\right)$$

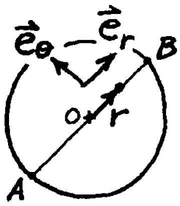
$$\boxed{|\theta| = 40.0^\circ}$$

2.3.2

GOAL: Determine the absolute speed and acceleration of a person running across a carousel when he reaches the edge.

GIVEN: The carousel is rotating at 0.6 rad/s (counter-clockwise) and its radius is 12 m. When Bill reaches the center O he has a speed of 1.5 m/s and an acceleration 0.6 m/s^2 ($t = 0$ is referenced to the point at which he reaches the center). At $t = 0$ the carousel begins to decelerate at a constant rate of 0.1 rad/s^2.

DRAW



FORMULATE EQUATIONS:

We'll use our expressions for angular velocity and acceleration

$$\vec{v}_A = \dot{r}\,\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

$$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \tag{2}$$

and our position, speed, constant acceleration relationships

$$a(t) = a_0, \quad v(t) = v_0 + a_0 t, \quad x(t) = x_0 + v_0 t + 0.5 a_0 t^2$$

SOLVE: First we solve for the elapsed time for Bill to go from O to B (radial motion):

$$12\,\text{m} = (1.5\,\text{m/s})t + 0.5(0.6\,\text{m/s}^2)t^2$$

which has solution $t = 4.30\,\text{s}$. Evaluating the radial speed corresponding to this time gives us $\dot{r} = 1.5\,\text{m/s} + (0.6\,\text{m/s}^2)(4.30\,\text{s}) = 4.08\,\text{m/s}$. At $t = 4.30\,\text{s}$ the carousel is rotating at $0.6\,\text{rad/s} - (0.1\,\text{rad/s}^2)(4.30\,\text{s}) = 0.170\,\text{rad/s}$.

(1) $\Rightarrow$
$$\vec{v}_{Bill} = [4.08\vec{e}_r + (12\,\text{m})(0.170\,\text{rad/s})\vec{e}_\theta]\,\text{m/s}$$

$$\boxed{\vec{v}_{Bill} = (4.08\vec{e}_r + 2.04\vec{e}_\theta)\,\text{m/s}}$$

(2) $\Rightarrow$

$\vec{a}_{Bill} = [(0.6\,\mathrm{m/s^2} - (12\,\mathrm{m})(0.170\,\mathrm{rad/s})^2)\vec{e}_r + ((12\,\mathrm{m})(-0.1\,\mathrm{rad/s^2}) + 2(4.08\,\mathrm{m/s})(0.170\,\mathrm{rad/s}))\vec{e}_\theta]\,\mathrm{m/s^2}$

$$\boxed{\vec{a}_{Bill} = (0.2532\,\vec{e}_r + 0.1872\,\vec{e}_\theta)\,\mathrm{m/s^2}}$$

2.3.11

GOAL: Determine if $\frac{d}{dt}(\overrightarrow{r})$ is zero.
GIVEN: $\dot{r} = 0.9$ m/s and $r\dot{\theta} = -0.9$ m/s.

FORMULATE EQUATIONS:

We'll use $\overrightarrow{v} = \dot{r}\overrightarrow{e}_r + r\dot{\theta}\overrightarrow{e}_\theta$.

SOLVE:

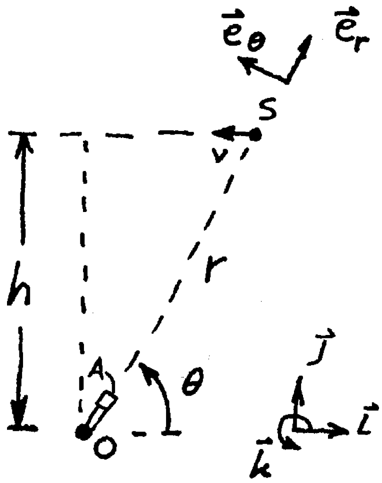
Although $\dot{r}$ and $r\dot{\theta}$ are the magnitudes of the components of $\frac{d}{dt}(\overrightarrow{r})$, these components are in orthogonal directions. $\frac{d}{dt}\overrightarrow{r}$ would equal $(0.9\overrightarrow{e}_r - 0.9\overrightarrow{e}_\theta)$ m/s, with a resultant magnitude of $0.9\sqrt{2}$ m/s. Thus the final answer to the question is "no."

2.3.13

GOAL: Determine the angular speed of rotation needed to keep the image of a bird centered in a camera's viewfinder and calculate the total acceleration of the far end of the lens.

GIVEN: Distance to the end of the lens from the point of rotation is 43 cm. The viewfinder displays a 102 cm wide target when the target is 24 m away. The swallow moves at a constant 64 km/h along a straight line. The closest approach of the swallow to the camera is $h = 24\,\mathrm{m}$.

DRAW:



FORMULATE EQUATIONS: The equations for velocity and acceleration in polar coordinates are

$$\vec{v}_S = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta \tag{1}$$

and

$$\vec{a}_S = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \tag{2}$$

SOLVE: 64 km/h is equivalent to 17.78 m/s. At $\theta = 90°$ $\vec{e}_r = \vec{j}$ and $\vec{e}_\theta = -\vec{i}$. Using the provided parameters in (1) gives us

$$\vec{v}_S = -17.78\,\vec{i}\ \mathrm{m/s} = \left[\dot{r}\,\vec{e}_r + (24\,\mathrm{m})\dot{\theta}\,\vec{e}_\theta\right]$$

$$\dot{r} = 0, \quad \dot{\theta} = \frac{17.78\,\mathrm{m/s}}{24\,\mathrm{m}} = 0.741\,\mathrm{rad/s}$$

$$\boxed{\dot{\theta} = 0.741\,\mathrm{rad/s}}$$

Using this data along with the provided parameters in (2) yields

$$\vec{a}_S = 0 = \left[\ddot{r} - (24\,\mathrm{m})(0.741\,\mathrm{rad/s})^2\right]\vec{e}_r + \left[2(0)(0.741\,\mathrm{rad/s}) + (24\,\mathrm{m})\ddot{\theta}\right]\vec{e}_\theta$$

$$\ddot{r} = 13.18\,\mathrm{m/s^2}, \quad \ddot{\theta} = 0$$

Our equation for the acceleration of end of the lens is a bit different than the swallow's acceleration because the swallow is moving in a straight line whereas the lens is rotating in a circle. Hence for the lens both $\dot{r}$ and $\ddot{r}$ are zero. The acceleration is found from

$$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta = -\left(\frac{43\,\mathrm{cm}}{100\,\mathrm{cm/\,m}}\right)(0.741\,\mathrm{rad/s})^2 + 0 = -0.236\vec{j}\ \mathrm{m/s^2}$$

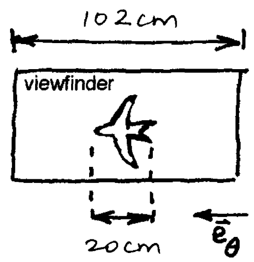
$$\boxed{\vec{a}_A = -0.236\vec{j}\ \mathrm{m/s^2}}$$

2.3.14

GOAL: Determine the constant angular velocity of the camera that will cause the image of a swallow to disappear from the viewfinder in 0.5 s. Compare this angular speed to the angular speed needed to perfectly track the bird and determine the percent variation from the ideal case.

GIVEN: The camera's telephoto lens captures 102 cm of target image when the object is 24 m away. The swallow is 20 cm long and initially centered. The bird is at $\theta = 90°$ and traveling at a constant 64 km/h with a constant radius of 24 m from the camera.

DRAW:



FORMULATE EQUATIONS: The equation for velocity in polar coordinates is

$$\vec{v}_m = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta \tag{1}$$

SOLVE: Using the provided parameters in (1) gives us a nominal rotation rate:

$$\vec{v}_S = (64\,\text{km/h})(0.2778\,\text{m/s})\vec{e}_\theta = 17.78\,\vec{e}_\theta\,\text{m/s} = (24\,\text{m})\dot{\theta}\,\vec{e}_\theta \;\Rightarrow\; \dot{\theta}_{\text{nominal}} = 0.741\,\text{rad/s}$$

If the camera is being rotated too slowly, the bird's image will move to the left in the viewfinder. It's initially centered and so in order to completely disappear from the viewfinder it needs to move $40\,\text{cm} + 20\,\text{cm} = 60\,\text{cm}$. The time for this to occur is given as 0.5 s. This implies a speed of $60\,\text{cm}/(0.5\,\text{s}) = 1.2\,\text{m/s}$. The corresponding angular speed is

$$\dot{\theta} = \frac{1.2\,\text{m/s}}{24\,\text{m}} = 0.05\,\text{rad/s}$$

Thus the angular rate of the camera would be the nominal rate $\pm$ this variation. Assuming a camera pan that's too slow (the usual problem) we have $\dot{\theta} = 0.741\,\text{rad/s} - 0.05\,\text{rad/s} = 0.691\,\text{rad/s}$.

$$\boxed{\dot{\theta} = 0.683\,\text{rad/s}}$$

The percent variation from nominal is given by $100\left(\frac{0.05}{0.741}\right)$:

$$\boxed{\text{percent variation} = 6.75\%}$$

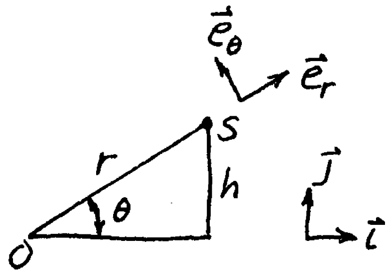
The implication is that it's not easy to keep a quickly moving object in the viewfinder, an observation that's very much supported by experience in the field.

2.3.15

GOAL: Calculate at what angle θ the camera's angular acceleration is a maximum and determine its angular acceleration at $\theta = 0$ and $\theta = 90°$.

GIVEN: The swallow moves at a constant 64 km/h along a straight line. The closest approach of the swallow to the camera is $h = 24\,\text{m}$.

DRAW:



	$\vec{i}$	$\vec{j}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS: The equations for velocity and acceleration in polar coordinates are

$$\vec{v}_S = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta \tag{1}$$

and

$$\vec{a}_S = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \tag{2}$$

SOLVE:

(1) $\Rightarrow$
$$-v\,\vec{i} = -v[\cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta] = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

$\vec{e}_r$:
$$\dot{r} = -v\cos\theta \tag{3}$$

$\vec{e}_\theta$:
$$\dot{\theta} = \frac{v\sin\theta}{r} \tag{4}$$

(2) $\Rightarrow$
$$0 = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta$$

$\vec{e}_r$:
$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{5}$$

$\vec{e}_\theta$:
$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{6}$$

$(3),(4),(6) \Rightarrow$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} = \frac{2v^2 \sin\theta \cos\theta}{r^2}$$

From geometry we see that $r = \frac{h}{\sin\theta}$ and hence we have

$$\ddot{\theta} = \frac{2v^2 \sin^3\theta \cos\theta}{h^2} \tag{7}$$

We find where $\ddot{\theta}$ is maximized (or minimized) by setting $\frac{d\,\ddot{\theta}}{d\,\theta} = 0$:

$$\frac{d\,\ddot{\theta}}{d\,\theta} = \frac{6\sin^2\theta\cos^2\theta - 2\sin^4\theta}{h^2} = 0$$

$$3\cos^2\theta = \sin^2\theta \;\Rightarrow\; \tan^2\theta = 3 \;\Rightarrow\; \tan\theta = \sqrt{3}$$

$$\boxed{\theta = 60^\circ}$$

Evaluating (7) at $\theta = 0$ gives us the same answer as for $\theta = 90^\circ$:

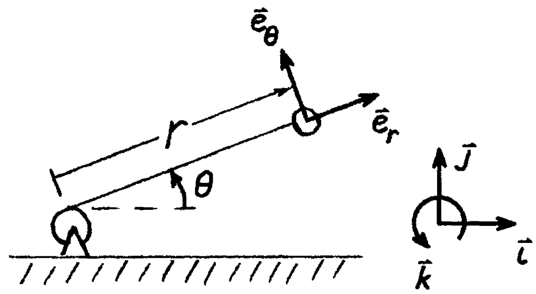
$$\boxed{\ddot{\theta} = 0}$$

2.3.17

GOAL: Determine $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$.

GIVEN: System geometry, and the exerciser's foot velocity and acceleration: $r = 0.6\,\text{m}$, $\theta = 15°$, $\vec{v} = 1.32\vec{i} + 0.76\vec{j}$ m/s, $\vec{a} = 0.27\vec{i} + 0.15\vec{j}$ m/s^2.

DRAW:



FORMULATE EQUATIONS:

The coordinate transformation array is

	$\vec{i}$	$\vec{j}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

The velocity of the exerciser's foot in terms of polar coordinates is

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

The acceleration of the exerciser's foot in terms of polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta \tag{2}$$

SOLVE:

To find $\dot{r}$ and $\dot{\theta}$,

(1) $\Rightarrow$

$$v_x\vec{i} + v_y\vec{j} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$v_x(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta) + v_y(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta) = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$\vec{e}_r$:

$$\dot{r} = v_x\cos\theta + v_y\sin\theta$$

$$\dot{r} = (1.32\,\text{m/s})\cos(15°) + (0.76\,\text{m/s})\sin(15°)$$

$$\boxed{\dot{r} = 1.472 \text{ m/s}} \tag{3}$$

$\vec{e}_\theta$:

$$r\dot{\theta} = -v_x \sin\theta + v_y \cos\theta$$

$$\dot{\theta} = \frac{1}{r}[-v_x \sin\theta + v_y \cos\theta]$$

$$\dot{\theta} = \frac{1}{0.6 \text{ m}}[-(1.32 \text{ m/s})\sin(15°) + (0.76\,\text{m/s})\cos(15°)]$$

$$\boxed{\dot{\theta} = 0.654\,\text{rad/s}} \tag{4}$$

To find $\ddot{r}$ and $\ddot{\theta}$,

(2) $\Rightarrow$

$$a_x\vec{i} + a_y\vec{j} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$$a_x(\cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta) + a_y(\sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta) = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$\vec{e}_r$:

$$\ddot{r} - r\dot{\theta}^2 = a_x \cos\theta + a_y \sin\theta$$

$$\ddot{r} = r\dot{\theta}^2 + a_x \cos\theta + a_y \sin\theta \tag{5}$$

(4) $\rightarrow$ (5) $\Rightarrow$

$$\ddot{r} = (0.6 \text{ m})(0.654\,\text{rad/s})^2 + (0.27\,\text{m/s}^2)\cos(15°) + (0.15\,\text{m/s}^2)\sin(15°)$$

$$\boxed{\ddot{r} = 0.692 \text{ m/s}^2}$$

$\vec{e}_\theta$:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -a_x \sin\theta + a_y \cos\theta$$

$$\ddot{\theta} = \frac{1}{r}[-2\dot{r}\dot{\theta} - a_x \sin\theta + a_y \cos\theta] \tag{6}$$

(3), (4) $\rightarrow$ (6) $\Rightarrow$

$$\begin{aligned}\ddot{\theta} &= \frac{1}{0.6 \text{ m}}[-2(1.472 \text{ m/s})(0.654\,\text{rad/s}) - (0.27 \text{ m/s}^2)\sin(15°) \\ &\quad + (0.15 \text{ m/s}^2)\cos(15°)]\end{aligned}$$

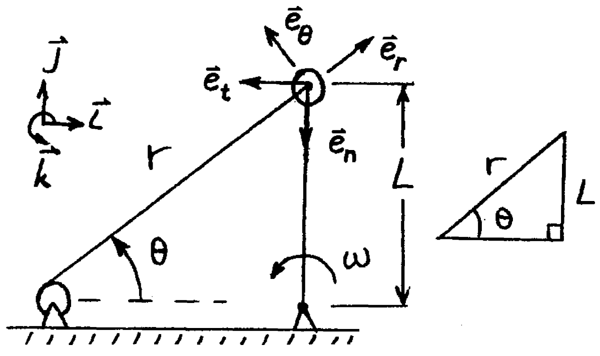
$$\boxed{\ddot{\theta} = -3.084\,\text{rad/s}^2}$$

2.3.19

GOAL: Determine r, $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$.

GIVEN: System geometry and the pole's constant angular speed: $L = 6\text{m}$, $\theta = 40°$, $\omega = 0.25\,\text{rad/s}$.

DRAW:



ASSUME: Neglect the mass of the pole, and analyze only the lumped mass on the end.

FORMULATE EQUATIONS:

The coordinate transformation array is

	$\vec{i}$	$\vec{j}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

With respect to the pole, the lumped mass's velocity and acceleration are, respectively,

$$\vec{v} = L\omega\vec{e}_t = -L\omega\vec{i} \tag{1}$$
$$\vec{a} = L\omega^2\vec{e}_n = -L\omega^2\vec{j} \tag{2}$$

The velocity and acceleration of the lumped mass in terms of polar coordinates are, respectively,

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{3}$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta \tag{4}$$

SOLVE:

From the diagram we see that

$$\begin{aligned} L &= r\sin\theta \\ r &= \frac{L}{\sin\theta} \\ r &= \frac{6\text{ m}}{\sin(40°)} \end{aligned}$$

$$\boxed{r = 9.33\ \text{m}} \tag{5}$$

To find $\dot{r}$ and $\dot{\theta}$,

(1) = (3) ⇒

$$-L\omega\vec{i} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$-L\omega(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta) = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$\vec{e}_r$:

$$\dot{r} = -L\omega\cos\theta$$

$$\dot{r} = -(6\ \text{m})(0.25\,\text{rad/s})\cos(40°)$$

$$\boxed{\dot{r} = -1.149\ \text{m/s}} \tag{6}$$

$\vec{e}_\theta$:

$$r\dot{\theta} = L\omega\sin\theta$$

$$\dot{\theta} = \frac{L\omega}{r}\sin\theta \tag{7}$$

(5) → (7) ⇒

$$\dot{\theta} = \frac{(6\ \text{m})(0.25\,\text{rad/s})}{9.33\ \text{m}}\sin(40°)$$

$$\boxed{\dot{\theta} = 0.103\,\text{rad/s}} \tag{8}$$

To find $\ddot{r}$ and $\ddot{\theta}$,

(2) = (4) ⇒

$$-L\omega^2\vec{j} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$$-L\omega^2(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta) = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$\vec{e}_r$:

$$\ddot{r} - r\dot{\theta}^2 = -L\omega^2\sin\theta$$

$$\ddot{r} = r\dot{\theta}^2 - L\omega^2\sin\theta \tag{9}$$

(5), (8) → (9) ⇒

$$\ddot{r} = (9.33\ \text{m})(0.103\,\text{rad/s})^2 - (6\ \text{m})(0.25\,\text{rad/s})^2\sin(40°)$$

$$\boxed{\ddot{r} = -0.142\ \text{m/s}^2}$$

$\vec{e}_\theta$:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -L\omega^2\cos\theta$$

$$\ddot{\theta} = -\frac{1}{r}[2\dot{r}\dot{\theta} + L\omega^2\cos\theta] \tag{10}$$

(5), (6), (8) → (10) ⇒

$$\begin{aligned}\ddot{\theta} &= -\frac{1}{9.33\ \text{m}}[2(-1.149\ \text{m/s})(0.103\,\text{rad/s}) \\ &\quad + (6\ \text{m})(0.25\,\text{rad/s})^2\cos(40°)]\end{aligned}$$

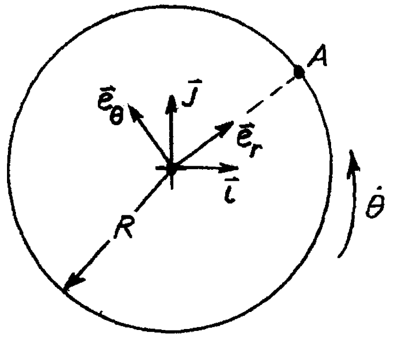
$$\boxed{\ddot{\theta} = -5.42\times10^{-3}\,\text{rad/s}^2}$$

2.3.20

GOAL: Determine $\|\vec{v}_A\|$ and $\|\vec{a}_A\|$.

GIVEN: The merry-go-round has radius $R = 1.5$m and is spun to a constant $\dot{\theta} = 3\,\text{rad/s}$. The RC toy car is accelerated from rest at the center straight out to point A at a rate $\ddot{r} = 0.9\ \text{m/s}^2$.

DRAW:



FORMULATE EQUATIONS:

With respect to the rotating merry-go-round, the RC toy car's velocity at point A is given by

$$\dot{r}_A^2 = 2\ddot{r}R \tag{1}$$

The velocity of the RC car at A in terms of polar coordinates is

$$\vec{v}_A = \dot{r}_A\vec{e}_r + R\dot{\theta}\vec{e}_\theta \tag{2}$$

The acceleration of the RC car at A in terms of polar coordinates is

$$\vec{a}_A = (\ddot{r} - R\dot{\theta}^2)\vec{e}_r + (2\dot{r}_A\dot{\theta})\vec{e}_\theta \tag{3}$$

SOLVE:

(2) $\Rightarrow$

$$\|\vec{v}_A\| = \sqrt{\vec{v}_A \cdot \vec{v}_A} = \sqrt{\dot{r}_A^2 + R^2\dot{\theta}^2} \tag{4}$$

(1) $\rightarrow$ (4) $\Rightarrow$

$$\|\vec{v}_A\| = \sqrt{2\ddot{r}R + R^2\dot{\theta}^2}$$

$$\|\vec{v}_A\| = \sqrt{2(0.9\ \text{m/s}^2)(1.5\ \text{m}) + (1.5\ \text{m})^2(3\,\text{rad/s})^2}$$

$$\boxed{\|\vec{v}_A\| = 4.79\ \text{m/s}}$$

(3) $\Rightarrow$
$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \sqrt{(\ddot{r} - R\dot{\theta}^2)^2 + 4\dot{r}_A^2\dot{\theta}^2} \tag{5}$$

(1) $\rightarrow$ (5) $\Rightarrow$
$$\|\vec{a}_A\| = \sqrt{(\ddot{r} - R\dot{\theta}^2)^2 + 8\ddot{r}R\dot{\theta}^2}$$

$$\|\vec{a}_A\| = \sqrt{\Big((0.9\ \mathrm{m/s^2}) - (1.5\ \mathrm{m})(3\,\mathrm{rad/s})^2\Big)^2 + 8(0.9\ \mathrm{m/s^2})(1.5\ \mathrm{m})(3\,\mathrm{rad/s})^2}$$

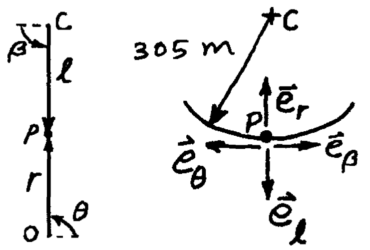
$$\boxed{\|\vec{a}_A\| = 15.999\ \mathrm{m/s^2}}$$

2.3.24

GOAL: Determine $\ddot{r}$ for a plane in a circular loop.

GIVEN: Path of plane, position of plane and observer and speed/acceleration information.

DRAW



FORMULATE EQUATIONS: Define the velocity and acceleration in terms of the $\vec{e}_l$, $\vec{e}_\beta$ unit vectors, referenced to the center of the loop C:

$$\vec{v}_P = \dot{l}\vec{e}_l + l\dot{\beta}\vec{e}_\beta$$

$$\vec{a}_P = (\ddot{l} - l\dot{\theta}^2)\vec{e}_l + (l\ddot{\beta} + 2\dot{l}\dot{\beta})\vec{e}_\beta \tag{1}$$

as well as the $\vec{e}_r$, $\vec{e}_\theta$ unit vectors, referenced to the ground O:

$$\vec{v}_P = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a}_P = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \tag{2}$$

ASSUME: We're given the fact that the path is circular and so can deduce that

$$\dot{l} = \ddot{l} = 0 \tag{3}$$

SOLVE:

$(1),(2) \Rightarrow$
$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta = (\ddot{l} - l\dot{\theta}^2)\vec{e}_l + (l\ddot{\beta} + 2\dot{l}\dot{\beta})\vec{e}_\beta \tag{4}$$

$(3),(4) \Rightarrow$
$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta = -l\dot{\theta}^2\vec{e}_l + l\ddot{\beta}\vec{e}_\beta$$

Realizing that for the configuration under consideration $\vec{e}_\theta = -\vec{e}_\beta$ and $\vec{e}_r = -\vec{e}_l$ gives us

$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta = l\dot{\theta}^2\vec{e}_r - l\ddot{\beta}\vec{e}_\theta$$

We're already given the fact that the acceleration is equal to 58.8 m/s^2 in the $\vec{e}_r$ direction and thus have

$$\ddot{r} - r\dot{\theta}^2 = 58.8\,\mathrm{m/s^2}$$

$$\ddot{r} = r\dot{\theta}^2 + 58.8\,\mathrm{m/s^2}$$

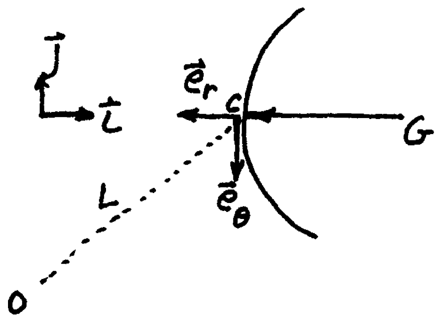
Both r and $\dot{\theta}$ are clearly non-zero and thus $\boxed{\ddot{r} \neq 58.8\,\mathrm{m/s^2}}$

2.3.27

GOAL: Find a_c and $\ddot{L}$.

GIVEN: v_t, $\dot{v}_t$ and car's position.

DRAW:



	$\vec{i}$	$\vec{j}$
$\vec{e}_r$	-1	0
$\vec{e}_\theta$	0	-1

FORMULATE EQUATIONS: The acceleration vector is

$$\vec{a}_c = (\ddot{r} - r\dot{\theta}^2)\,\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\vec{e}_\theta$$

$$r\dot{\theta} = v \quad \Rightarrow \quad \dot{\theta} = \frac{v}{r} = \frac{(80.5 \text{ km/h})(\frac{1000}{3600})}{90 \text{ m}} = 0.248 \text{ rad/s}$$

and

$$r\ddot{\theta} = 0.3(9.81 \text{ m/s}^2) \quad \Rightarrow \quad \ddot{\theta} = \frac{0.3(9.81)}{90} = 0.0327 \text{ rad/s}^2$$

also

$$\dot{r} = \ddot{r} = 0$$

SOLVE: Calculate the acceleration

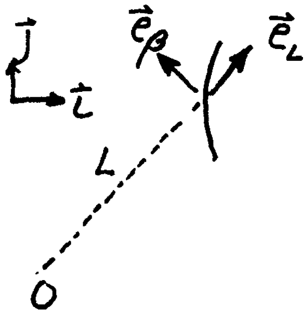
$$\vec{a}_c = -(90 \text{ m})(0.248 \text{ rad/s})^2\,\vec{e}_r + (90 \text{ m})(0.0327 \text{ rad/s}^2)\,\vec{e}_\theta$$

$$\boxed{\vec{a}_c = (-5.535\,\vec{e}_r + 2.94\,\vec{e}_\theta) \text{ m/s}^2 = (5.535\,\vec{i} - 2.94\,\vec{j}\,) \text{ m/s}^2}$$

	$\vec{i}$	$\vec{j}$
$\vec{e}_r$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\vec{e}_\theta$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

Next we'll find $\ddot{L}$. The velocity vector $\vec{v}_c$ is

$$\vec{v}_c = \dot{L}\,\vec{e}_L + L\dot{\beta}\,\vec{e}_\beta = -v\,\vec{j}\ = -v(\frac{1}{\sqrt{2}}\,\vec{e}_L + \frac{1}{\sqrt{2}}\,\vec{e}_\beta)$$



$\vec{e}_L$:
$$\dot{L} = -\frac{v}{\sqrt{2}} = -\frac{22.36\text{ m/s}}{\sqrt{2}} = -15.81\text{ m/s}$$

$\vec{e}_\beta$:
$$\dot{\beta} = -\frac{v}{\sqrt{2}(L)} = -\frac{23.36\text{ m/s}}{\sqrt{2}(\sqrt{2}(90\text{ m}))} = -0.1298\text{ rad/s}$$

The acceleration vector $\vec{a}_c$ is

$$\vec{a}_c = (\ddot{L} - L\dot{\beta}^2)\,\vec{e}_L + (L\ddot{\beta} + 2\dot{L}\dot{\beta})\,\vec{e}_\beta = (-5.535\,\vec{i}\ - 2.94\,\vec{j}\ )\text{ m/s}^2$$

$\vec{e}_L$:
$$\ddot{L} - L\dot{\beta}^2 = \left(-\frac{2.94}{\sqrt{2}} + \frac{5.535}{\sqrt{2}}\right)\text{ m/s}^2$$

$\vec{e}_\beta$:
$$L\ddot{\beta} + 2\dot{L}\dot{\beta} = \left(-\frac{2.94}{\sqrt{2}} - \frac{5.535}{\sqrt{2}}\right)\text{ m/s}^2$$

SOLVE: Solve for $\ddot{L}$

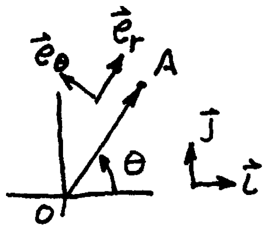
$$\boxed{\ddot{L} = \sqrt{2}(90\text{ m})(0.1298\text{ rad/s})^2 - \tfrac{1}{\sqrt{2}}(2.94 - 5.535)\text{ m/s}^2 = 3.98\text{ m/s}^2}$$

2.3.28

GOAL: Find velocity and acceleration functions of time that describe the motion of point A

GIVEN: $r = a\theta$, $a = 3.05$ m/rad $\dot{\theta} = 10$ rad/s

DRAW:



ASSUME: $\ddot{\theta} = 0$

FORMULATE EQUATIONS:

polar velocity: $$\vec{v}_A = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

polar acceleration: $$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

SOLVE:

$$\theta = \int \dot{\theta} dt = 10 \text{ rad/s} \int dt = (10 \text{ rad/s})t$$

$$r = a\theta = (3.05 \text{ m/rad})[(10 \text{ rad/s})t] = (30.5 \text{ m/s})t$$

$$\dot{r} = 30.5 \text{ m/s} \quad \ddot{r} = 0$$

polar velocity $$\vec{v}_A = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta = (30.5 \text{ m/s})\vec{e}_r + (30.5 \text{ m/s})t(10 \text{ rad/s})\vec{e}_\theta$$

$$\boxed{\vec{v}_A = (30.5 \text{ m/s})\vec{e}_r + (305 \text{ m/s}^2)t\,\vec{e}_\theta}$$

polar accel $$\vec{a}_A = (0 - r\dot{\theta}^2)\vec{e}_r + (0 + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

$$\vec{a}_A = [-(30.5 \text{ m/s})t(10 \text{ rad/s})^2]\vec{e}_r + (2)(30.5 \text{ m/s})(10 \text{ rad/s})\vec{e}_\theta$$

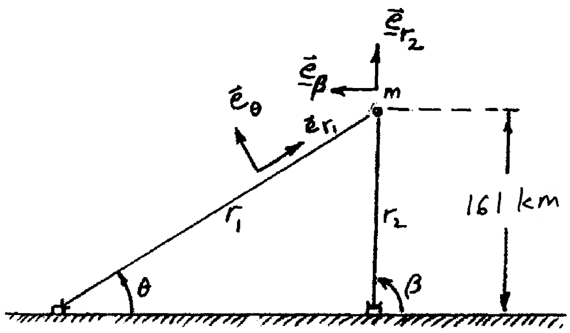
$$\boxed{\vec{a}_A = -(3050 \text{ m/s}^3)t\,\vec{e}_r + (610 \text{ m/s}^2)\vec{e}_\theta}$$

2.3.34

GOAL: Determine kinematic data of a satellite from ground information.

GIVEN: Position and assorted kinematic data.

DRAW:



	$\vec{e}_{r_1}$	$\vec{e}_\theta$
$\vec{e}_{r_2}$	$\cos(\theta-\beta)$	$-\sin(\theta-\beta)$
$\vec{e}_\beta$	$\sin(\theta-\beta)$	$\cos(\theta-\beta)$

FORMULATE EQUATIONS:

We'll be using the general formulas for velocity and acceleration in a polar frame, expressed in terms of the two sets of unit vectors:

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta$$

SOLVE:

Using the provided values gives

$$\vec{a} = \left(\ddot{r}_2 - \left(\frac{1000\text{ m}}{1\text{ km}}\right)(161\text{ km})(-4.88\times10^{-2}\text{ rad/s})^2\right)\vec{e}_{r_2} + (1000\,/\text{km})(161\text{ km})\ddot{\beta}\vec{e}_\beta$$

$$\vec{a} = \left(\ddot{r}_2 - 383\text{ m/s}^2\right)\vec{e}_{r_2} + \left(1.61\times10^5\text{ m}\right)\ddot{\beta}\vec{e}_\beta$$

The acceleration of the satellite is

$$\vec{a} = -9.36\vec{e}_{r_2}\text{ m/s}^2$$

so

$$-9.36\vec{e}_{r_2}\text{ m/s}^2 = \vec{a} = \left(\ddot{r}_2 - 383\text{ m/s}^2\right)\vec{e}_{r_2} + \left(1.61\times10^5\text{ m}\right)\ddot{\beta}\vec{e}_\beta$$

Equating coefficients gives for $\vec{e}_{r2}$

$$\ddot{r}_2 - 383 \text{ m/s}^2 = -9.36 \text{ m/s}^2$$

$$\boxed{\ddot{r}_2 = 373.6\,\text{m/s}^2}$$

and for $\vec{e}_\beta$

$$1.61 \times 10^5 \ddot{\beta} = 0$$

$$\boxed{\ddot{\beta} = 0}$$

The first step to solve for the values with respect to Station A is to find the velocity of the satellite with respect to Station B. The velocity is

$$\vec{v}_B = \dot{r}_2 \vec{e}_{r_2} + r_2 \dot{\theta} \vec{e}_\beta$$

$$\vec{v}_B = 0\vec{e}_{r_2} + (1.61 \times 10^5 \text{ ft})(-4.88 \times 10^{-2} \text{ rad/s})\vec{e}_\beta$$

$$\vec{v}_B = -7.857 \times 10^3 \vec{e}_\beta \text{ m/s}$$

Applying a coordinate transform with $\theta = 45°$, $\beta = 90°$ gives

$$\vec{v}_A = (-7.857 \times 10^3 \text{ m/s}) \left(-\frac{\sqrt{2}}{2}\vec{e}_{r_1} + \frac{\sqrt{2}}{2}\vec{e}_\theta \right)$$

So

$$\vec{v}_A = [5.56 \times 10^3 \vec{e}_{r_1} - 5.56 \times 10^3 \vec{e}_\theta] \text{ m/s}$$

Now set this equation equal to the general equation for velocity in polar coordinates

$$\dot{r}_1 \vec{e}_{r_1} + r_1 \dot{\theta} \vec{e}_\theta = [5.56 \times 10^3 \vec{e}_{r_1} - 5.56 \times 10^3 \vec{e}_\theta] \text{ m/s} \tag{1}$$

Use the station's geometry to solve for r_1

$$r_1 = \frac{161 \text{ km}}{\sin 45^o}$$

So

$$\boxed{r_1 = 2.28 \times 10^5\,\text{m}}$$

Equating coefficients in (1) gives for $\vec{e}_{r_1}$

$$\boxed{\dot{r}_1 = 5.56 \times 10^3\,\text{m/s}}$$

and for $\vec{e}_\theta$

$$r_1 \dot{\theta} = -5.56 \times 10^3 \text{ m/s}$$

So

$$\boxed{\dot{\theta} = -2.44 \times 10^{-2}\,\text{rad/s}}$$

Now, to solve for the rest of the variables, write the acceleration equations

$$\vec{a}_B = -9.36 \vec{e}_{r_2} \text{ m/s}^2$$

Transforming to station A's coordinates

$$\vec{a}_A = (-6.62\vec{e}_{r_1} - 6.62\vec{e}_\theta) \text{ m/s}^2$$

Writing the general equation

$$\vec{a}_A = \left(\ddot{r}_1 - r_1\dot{\theta}^2\right)\vec{e}_{r_1} + \left(2\dot{r}_1\dot{\theta} + r_1\ddot{\theta}\right)\vec{e}_\theta$$

and equating coefficients for $\vec{e}_{r_1}$ gives

$$-6.62 \text{ m/s}^2 = \ddot{r}_1 - (2.28 \times 10^5 \text{ ft})(-2.44 \times 10^{-2} \text{ rad/s})^2$$

$$\boxed{\ddot{r}_1 = 129.1\,\text{m/s}^2}$$

Equating for $\vec{e}_\theta$ gives

$$-6.62 \text{ m/s}^2 = 2(5.56 \times 10^3 \text{ m/s})(-2.44 \times 10^{-2} \text{ rad/s}) + (2.28 \times 10^5 \text{ m})\ddot{\theta}$$

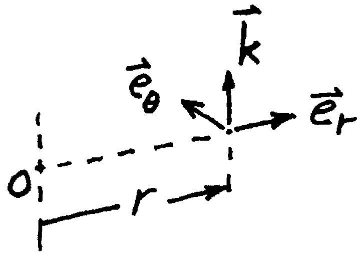
$$\boxed{\ddot{\theta} = 1.16 \times 10^{-3}\,\text{rad/s}^2}$$

2.3.37

GOAL: Find magnitude and direction of thief's velocity and acceleration.

GIVEN: Magnitude of thief's speed and dimensions of the museum floor.

DRAW:



FORMULATE EQUATIONS:

Velocity
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + \dot{h}\vec{e}_z \tag{1}$$

Acceleration
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta + \ddot{h}\vec{e}_z \tag{2}$$

The thief is constrained to remain on the floor,

Constraint
$$\dot{h} = h_0 - \frac{3.5\,\text{mt}}{2\pi}\theta \tag{3}$$

where, h_0 is the initial height (in m) and θ is the thief's angular position specified in radians.

Differentiating (3) twice,
$$\dot{h} = -\frac{3.5}{2\pi}\dot{\theta}\,\text{m/s} \tag{4}$$

$$\ddot{h} = -\frac{3.5\ \text{m}}{2\pi}\ddot{\theta}\,\text{m/s}^2 \tag{5}$$

SOLVE:

(4) → (1) ⇒
$$\vec{v} = r\dot{\theta}\vec{e}_\theta - \frac{3.5\,\text{m}}{2\pi}\dot{\theta}\vec{e}_z\,\text{m/s} \tag{6}$$

Speed is 20 km/h ⇒
$$\sqrt{\vec{v}\cdot\vec{v}} = \left(r^2\dot{\theta}^2 + \frac{12.25}{4\pi^2}\dot{\theta}^2\right)^{1/2} = 20\,\text{km/hr} \tag{7}$$

(7) ⇒
$$\dot{\theta} = 0.4625\,\text{rad/s} \tag{8}$$

$$\boxed{\vec{v} = 5.55\vec{e}_\theta - 0.258\vec{e}_z\ \text{m/s}}$$

$$\boxed{\text{direction of velocity} = \tan^{-1}\tfrac{0.258}{5.55} = 2.66^\circ}$$

(8), (5) $\rightarrow$ (2) $\Rightarrow$ $$\vec{a} = -12\dot{\theta}^2\vec{e}_r \,\mathrm{m/s^2}$$

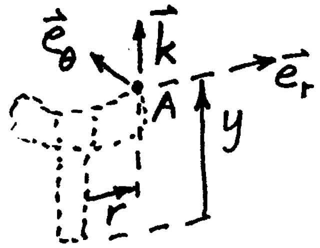
$$\boxed{\|\vec{a}\| = -2.57\,\mathrm{m/s^2},\ \vec{a} \text{ is in radial direction}}$$

2.3.38

GOAL: Find a) revolutions of wingnut in given time, b) percentage increase in speed due to moving down.

GIVEN: vertical speed, constant angular velocity, and total acceleration of fixed point on wingnut.

DRAW:



FORMULATE EQUATIONS:

Because wingnut has constant angular velocity, it must also have constant vertical velocity due to constant thread pitch:

Vertical velocity $$\dot{h} = \frac{2.3\,\text{cm}}{1.2\,\text{s}} \tag{1}$$

Acceleration $$\vec{a}_A = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta + \ddot{h}\,\vec{e}_z \tag{2}$$

SOLVE:

(2) $\Rightarrow$ $$\|\vec{a}_A\| = \| - r\dot{\theta}^2\vec{e}_r\| = 5.6\times 10^3\,\text{cm/s}^2 \tag{3}$$

(3) $\Rightarrow$ $$\dot{\theta} = 52.92\,\text{rad/s} \tag{4}$$

Integrating (4) $\Rightarrow$ $$\int_0^{1.2}\dot{\theta}dt = 52.92(1.2)\,\text{rad}$$

$$\boxed{\theta = 63.504\,\text{rad} = 10\,\text{revolutions}}$$

Speed at constant height $\Rightarrow$ $$\|\vec{v}_1\| = r\dot{\theta} \tag{5}$$

Speed at variable height $\Rightarrow$ $$\|\vec{v}_2\| = \sqrt{(r\dot{\theta})^2 + \dot{h}^2} \tag{6}$$

% increase in speed $\Rightarrow$ $$100\times\frac{\|\vec{v}_2\| - \|\vec{v}_1\|}{\|\vec{v}_1\|} \tag{7}$$

(5), (6) $\to$ (7) $\Rightarrow$ $$\frac{\sqrt{[(2\,\text{cm})(52.92\,\text{rad/s})]^2 + \left(\frac{2.3\,\text{cm}}{1.2\,\text{s}}\right)^2} - (2\,\text{cm})(52.92\,\text{rad/s})}{(2\,\text{cm})(52.92\,\text{rad/s})}$$

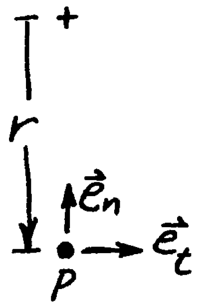
$$\boxed{\%\text{ increase in speed} = 0.016\,\%}$$

2.4.1

GOAL: Find the normal acceleration felt by the airplane if the pilot pulls into an upward loop.

GIVEN: The radius of the loop is 396 m.

DRAW:



FORMULATE EQUATIONS: We'll use the formula for acceleration in path coordinates:

$$\vec{a}_P = a_t \vec{e}_t + a_n \vec{e}_n = \dot{v} \vec{e}_t + \frac{v^2}{r_c} \vec{e}_n$$

SOLVE:

The radius of the loop is the radius of curvature for this problem:

$$a_n = \frac{v^2}{r} = \frac{(90\,\mathrm{m/s})^2}{396\mathrm{m}}$$

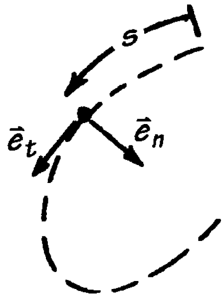
$$\boxed{a_n = 20.45\,\mathrm{m/s^2}}$$

2.4.5

GOAL: Determine the constant $v_{\max}$ before the motorcycle begins to slip at $s = 6\,\mathrm{m}$.

GIVEN: The maximum allowable acceleration of the motorcycle before slip is $a_{\max} = 9\,\mathrm{m/s^2}$. The path's radius of curvature is described by $r_c = (7.62 - 6 \times 10^{-4} s^3)\,\mathrm{m}$. The tangential speed is constant.

DRAW:



FORMULATE EQUATIONS:

The acceleration of the motorcycle in terms of path coordinates is

$$\vec{a} = \frac{v^2}{r_c}\vec{e}_n \tag{1}$$

SOLVE:

(1) $\Rightarrow$

$$a_{\max} = \frac{v_{\max}^2}{r_c}$$

$$\begin{aligned} v_{\max} &= \sqrt{a_{\max} r_c} \\ v_{\max} &= \sqrt{(9\,\mathrm{m/s^2})\,\{7.62 - 6 \times 10^{-4}(20)^3\,\mathrm{m}\}} \end{aligned}$$

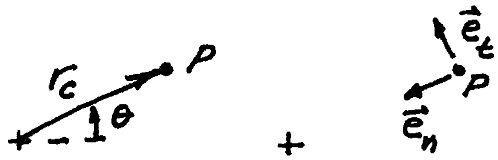
$$\boxed{v_{\max} = 5.04\,\mathrm{m/s}}$$

2.4.6

GOAL: Find the rotational speed of a Ferris wheel.

GIVEN: Magnitude of acceleration and dimensions.

DRAW:



FORMULATE EQUATIONS:

Velocity of P

$$\vec{v}_P = r_C\dot{\theta}\,\vec{e}_t \tag{1}$$

Acceleration of P

$$\vec{a}_P = a_t\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n \tag{2}$$

SOLVE:

constant rotational speed$\Rightarrow$

$$\vec{a}_P = \frac{v^2}{r_C}\vec{e}_n \tag{3}$$

(1) $\rightarrow$ (3) $\Rightarrow$

$$\vec{a}_P = \frac{r_C^2\dot{\theta}^2}{r_C} = 0.1\,\mathrm{m/s^2}$$

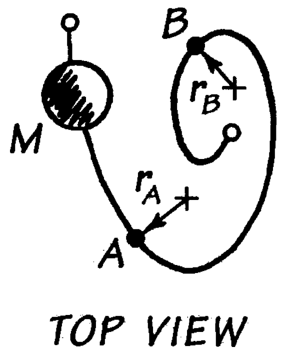
$$\boxed{\dot{\theta} = 0.105\,\mathrm{rad/s} = 1\,\mathrm{rev/min}}$$

2.4.8

GOAL: Determine the marble's constant speed v_A at A and its tangential acceleration $a_{B,t}$ at B.

GIVEN: At point A, the radius of curvature is $r_A = 0.2\,\text{m}$, and the marble's total acceleration is $\|\vec{a}_A\| = 4\,\text{m/s}^2$. At point B, the marble is traveling at $v_B = 0.8\,\text{m/s}$ with a total acceleration of $\|\vec{a}_B\| = 5.5\,\text{m/s}^2$. The radius of curvature at B is $r_B = 0.12\,\text{m}$.

DRAW:



FORMULATE EQUATIONS:

The marble's acceleration in terms of path coordinates is given by

$$\vec{a} = a_t\vec{e}_t + \frac{v_t^2}{r}\vec{e}_n \tag{1}$$

SOLVE:

We're told that the marble's speed at A is constant, and so its total acceleration is given by the acceleration normal to the path:

(1) $\Rightarrow$

$$\vec{a}_A = \frac{v_A^2}{r_A}\vec{e}_n$$

$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \frac{v_A^2}{r_A}$$

$$v_A = \sqrt{\|\vec{a}_A\| r_A}$$

$$v_A = \sqrt{(4\,\text{m/s}^2)(0.2\,\text{m})}$$

$$\boxed{v_A = 0.894\,\text{m/s}}$$

We can find the marble's tangential acceleration at B by rearranging our expression for its total acceleration at that point:

(1) $\Rightarrow$

$$\vec{a}_B = a_{B,t}\,\vec{e}_t + \frac{v_B^2}{r_B}\vec{e}_n$$

$$\|\vec{a}_B\|^2 = \vec{a}_B \cdot \vec{a}_B = (a_{B,t})^2 + \left(\frac{v_B^2}{r_B}\right)^2$$

$$a_{B,t} = \sqrt{\|\vec{a}_B\|^2 - \left(\frac{v_B^2}{r_B}\right)^2}$$

$$a_{B,t} = \sqrt{(5.5\,\mathrm{m/s^2})^2 - \left[\frac{(0.8\,\mathrm{m/s})^2}{0.12\,\mathrm{m}}\right]^2}$$

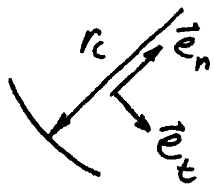
$$\boxed{a_{B,t} = 1.34\,\mathrm{m/s^2}}$$

2.4.9

GOAL: Find r_{min} to support a maximum normal acceleration of 3.5g

GIVEN: Path and speed.

DRAW:



FORMULATE EQUATIONS:

$$\vec{a} = \dot{v}\,\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

SOLVE:

$$96.5 \text{ km/h} = 26.8 \text{ m/s}$$

$$a_n = \frac{v^2}{r_C} = \frac{(26.8 \text{ m/s})^2}{r_C} = (3.5)(9.81 \text{ m/s}^2)$$

$$r_C = 20.92 \text{ m}$$

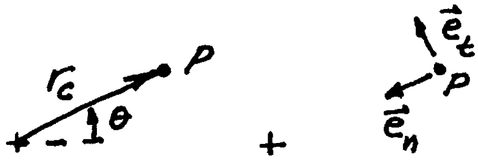
$$\boxed{r_{min} = r_C = 20.92 \text{ m}}$$

2.4.10

GOAL: Find acceleration of point P at $t = 3\,\text{s}$.

GIVEN: Tangential acceleration, initial conditions, and dimensions.

DRAW:



FORMULATE EQUATIONS:

$a_t = \frac{dv}{dt} = ct \Rightarrow$ $$dv = c\,t\,dt \Rightarrow \int_0^t dv = \int_0^t \tau d\tau \Rightarrow v(t) - v(0) = \frac{c\,t^2}{2} \tag{1}$$

Acceleration of P $$\vec{a}_P = a_t \vec{e}_t + \frac{v^2}{r_C}\vec{e}_n = ct\vec{e}_t + \frac{v(t)^2}{r_C}\vec{e}_n \tag{2}$$

SOLVE:

(1) and given conditions$\Rightarrow$ $$v(3\,\text{s}) = \frac{(-0.12\,\text{m/s}^3)(3\,\text{s})^2}{2} + 3\,\text{m/s} = 2.46\,\text{m/s} \tag{3}$$

(3) $\rightarrow$ (2) $\Rightarrow$

$$\vec{a}_P(3\,\text{s}) = \left[(-0.12\,\text{m/s}^3)(3\,\text{s})\vec{e}_t + \frac{(2.46\,\text{m/s})^2}{9\,\text{m}}\vec{e}_n\right]$$

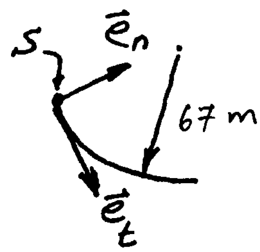
$$\boxed{\vec{a}_P(3\,\text{s}) = (-0.36\,\text{m}\,\vec{e}_t + 0.6724\vec{e}_n)\,\text{m/s}^2}$$

2.4.12

GOAL: Find the acceleration magnitude of a skier, S.

GIVEN: Skier's path and speed.

DRAW



FORMULATE EQUATIONS:

$$\vec{a} = \dot{v}\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

SOLVE: We need to find the skier's speed at B. The acceleration is constant and so we have

$$\frac{v^2}{2} = a\Delta s = (8.5 \text{ m/s}^2)(67 \text{ m}) = 569.5 \text{ (m/s)}^2$$

$$v = 33.75 \text{ m/s}$$

$$\vec{a}_S = 8.5 \text{ m/s}^2\vec{e}_t + \frac{(33.75 \text{ m/s})^2}{67 \text{ m}}\vec{e}_n = [8.5\vec{e}_t + 17.0\vec{e}_n] \text{ m/s}^2$$

$$\boxed{|\vec{a}_S| = 19.0 \text{ m/s}^2 = 1.937\,\text{g}}$$

2.4.13

GOAL: Find direction and magnitude of velocity of point b immediately before and after reaching point A.

GIVEN: Constraint of constant speed before reaching point A, acceleration after reaching point A.

DRAW:

FORMULATE EQUATIONS:

Velocity of point b before reaching point A

$$\vec{v}_{before} = \vec{v}_b = v_b \vec{\imath} \tag{1}$$

Velocity of point b after reaching point A

$$\vec{v}_{after} = \vec{v}_b + \int_{t_A}^{t} \vec{a} dt \tag{2}$$

SOLVE:

Immediately after passing point A, $t \to t_A$ in (2)$\Rightarrow$ $\vec{v}_{after} = \vec{v}_b + \int_{t_A}^{t_A} \vec{a} dt = \vec{v}_b$

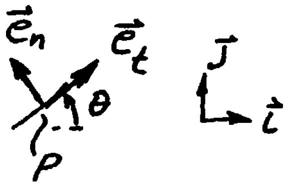
$$\boxed{\vec{v}_{before} = \vec{v}_{after} = v_b \vec{\imath}}$$

2.4.15

GOAL: Determine the normal and tangential acceleration of the point P when it has the same speed in both the $\vec{i}$ and $\vec{j}$ directions.

GIVEN: $\dot{y} = bx$, with $b = 12\,\mathrm{s}^{-1}$.

DRAW:



	$\vec{i}$	$\vec{j}$
$\vec{e}_t$	$\cos\theta$	$\sin\theta$
$\vec{e}_n$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS: We'll determine the acceleration in terms of $\vec{i}, \vec{j}$ and then transform this into the path frame.

SOLVE:

Differentiating $y = ax^2$ with respect to time gives us

$$\dot{y} = 2ax\dot{x} \tag{1}$$

$\dot{x}$ and $\dot{y}$ will be equal if $2ax = 1$. Hence we have

$$\boxed{x = \tfrac{1}{2a} = \tfrac{1}{20\,\mathrm{m}^{-1}} = \tfrac{1}{20}\,\mathrm{m}}$$

as the position at which $\dot{x}$ and $\dot{y}$ are equal. Furthermore, we're given

$$\dot{y} = bx \tag{2}$$

$$(1),(2) \Rightarrow \qquad 2ax\dot{x} = bx \Rightarrow \dot{x} = \frac{b}{2a} = \frac{12\,\mathrm{s}^{-1}}{20\,\mathrm{m}^{-1}} = 0.6\,\mathrm{m/s} \tag{3}$$

If $\dot{x}$ and $\dot{y}$ are equal the $\vec{e}_t$ unit vector is oriented up by $45°$ (since it points along the velocity vector).

$$(2) \Rightarrow \qquad \dot{y} = bx = (12\,\mathrm{s}^{-1})(\frac{1}{20}\,\mathrm{m}) = 0.6\,\mathrm{m/s} \tag{4}$$

and

$$\boxed{v = \sqrt{(0.6\,\mathrm{m/s})^2 + (0.6\,\mathrm{m/s})^2} = 0.85\,\mathrm{m/s}}$$

$$(1) \Rightarrow \qquad \ddot{y} = 2a\dot{x}^2 + 2ax\ddot{x} \tag{5}$$

$$(2) \Rightarrow \qquad \ddot{y} = b\dot{x} \tag{6}$$

$$(5),(6) \Rightarrow \qquad (20\,\mathrm{m}^{-1})(0.6\,\mathrm{m/s})^2 + \ddot{x} = (20\,\mathrm{m}^{-1})(0.6\,\mathrm{m/s})^2 \Rightarrow \ddot{x} = 0 \tag{7}$$

$$(6) \Rightarrow \qquad \ddot{y} = (12\,\mathrm{s}^{-1})(0.6\,\mathrm{m/s}) = 7.2\,\mathrm{m/s} \tag{8}$$

$$(7),(8) \Rightarrow \qquad a_t\left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right)\,\mathrm{m/s^2} + a_n\left(-\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right)\,\mathrm{m/s^2} = 7.2\vec{j}\ \mathrm{m/s^2} \tag{9}$$

$$\vec{i}\ : \qquad \frac{a_t}{\sqrt{2}} - \frac{a_n}{\sqrt{2}} = 0 \tag{10}$$

$$\vec{j}\ : \qquad \frac{a_t}{\sqrt{2}} + \frac{a_n}{\sqrt{2}} = 7.2\,\mathrm{m/s^2} \tag{11}$$

$$\boxed{a_t = 5.09\,\mathrm{m/s^2}}$$

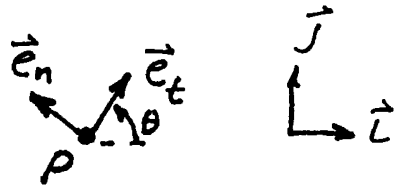
$$\boxed{a_n = 5.09\,\mathrm{m/s^2}}$$

2.4.16

GOAL: Determine the speed of a moving particle when it is at $x = \frac{\pi}{2\lambda}$.

GIVEN: $y = a[1 - \cos(\lambda x)]$ with $\lambda = 0.01\,\mathrm{m}^{-1}$ and $a = 24\,\mathrm{m}$. When at $x = \frac{\pi}{2\lambda}$, $a_P = (2.4\vec{\imath} + 0.6\vec{\jmath}\,)\,\mathrm{m/s^2}$.

DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_t$	$\cos\theta$	$\sin\theta$
$\vec{e}_n$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS: We'll determine the acceleration in terms of $\vec{\imath}, \vec{\jmath}$ and then transform this into the path frame using

$$a_t\vec{e}_t + a_n\vec{e}_n = a_x\vec{\imath} + a_y\vec{\jmath}$$

SOLVE: To find the angle θ we can calculate $\frac{dy}{dx}$ at $x = \frac{\pi}{2\lambda}$:

$$\frac{dy}{dx} = a\lambda\sin\left[\lambda\left(\frac{\pi}{2\lambda}\right)\right] = (24\,\mathrm{m})(0.01\,\mathrm{m}^{-1}) = 0.24$$

$$\frac{dy}{dx} = 0.24 \Rightarrow \theta = 14.0^\circ$$

$$(2.4\vec{\imath} + 0.6\vec{\jmath}\,)\,\mathrm{m/s^2} = a_t\vec{e}_t + a_n\vec{e}_n = a_t\left(0.970\vec{\imath} + 0.243\vec{\jmath}\,\right) + a_n\left(-0.243\vec{\imath} + 0.970\vec{\jmath}\,\right)$$

$\vec{\imath}$:
$$2.4\,\mathrm{m/s^2} = 0.970a_t - 0.243a_n$$

$\vec{\jmath}$:
$$0.6\,\mathrm{m/s^2} = 0.243a_t + 0.970a_n$$

Solving these two equations gives us $a_n = 0$. Since the normal acceleration is given by $a_n = \frac{v^2}{r_C}$ we can have a zero value of this acceleration component through having a speed of zero or an infinite radius of curvature. The radius of curvature at the give point is found from

$$r_C = \frac{\left[1 + (\frac{dy}{dx})^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$$

Evaluating the second derivative of y at $x = \frac{\pi}{2\lambda}$ gives us

$$\frac{d^2y}{dx^2} = a\lambda^2 \cos\left[\lambda\left(\frac{\pi}{2\lambda}\right)\right] = 0$$

Thus we see that our radius of curvature is infinite. We must therefore conclude that we *can't* determine v because any finite v will produce a zero normal acceleration component due to the infinite radius of curvature.

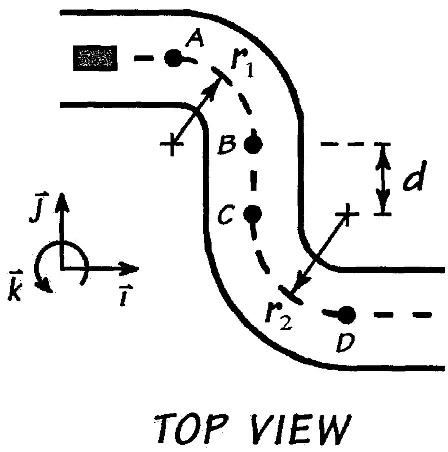
$$\boxed{v \text{ cannot be determined}}$$

2.4.17

GOAL: Determine the maximum constant acceleration in the straightaway from B to C so that the car just makes it through the bend from C to D without losing traction, and find the corresponding time to go from A to D.

GIVEN: From A to B, the car travels at a constant speed of $v_A = 40\,\text{km/hr}$ through a 90° arc of radius $r_1 = 21\,\text{m}$. The straightaway from B to C is $d = 12\,\text{m}$ long, and the bend from C to D is a 90° arc with a radius of $r_2 = 24\,\text{m}$. The car's speed is constant from C to D. The car will slide off the road when its normal acceleration exceeds $a_{n,\max} = 0.75g$.

DRAW:



FORMULATE EQUATIONS:

Since the car's speed is constant through both bends, its acceleration during the turns can be expressed in terms of path coordinates as

$$\vec{a} = \frac{v_t^2}{r}\vec{e}_n \tag{1}$$

The time it takes for the car to travel through each turn with a constant speed is governed by

$$v = \frac{\Delta s}{\Delta t} \tag{2}$$

When the car is accelerating at a constant rate in the straightaway, we can say that

$$v^2 - v_0^2 = 2a\Delta s \tag{3}$$

$$a = \frac{\Delta v}{\Delta t} \tag{4}$$

SOLVE:

The car will lose traction when its normal acceleration exceeds $a_{n,\max} = 0.75g = 7.36\,\mathrm{m/s^2}$, so let's first verify that it will make it through the first turn from A to B:

(1) $\Rightarrow$

$$\vec{a}_{AB} = \frac{v_A^2}{r_1}\vec{e}_n$$

$$\|\vec{a}_{AB}\| = \sqrt{\vec{a}_{AB} \cdot \vec{a}_{AB}} = \frac{v_A^2}{r_1}$$

$$\|\vec{a}_{AB}\| = a_{AB,n} = \frac{\left[(40\,\mathrm{km/hr})\left(\frac{\mathrm{hr}}{3600\,\mathrm{s}}\right)\left(\frac{1000\,\mathrm{m}}{\mathrm{km}}\right)\right]^2}{21\,\mathrm{m}}$$

$$a_{AB,n} = 5.88\,\mathrm{m/s^2} < a_{n,\max}$$

We find that the car indeed makes it through the first bend, and thus the time it takes to go from A to B is

(2) $\Rightarrow$

$$t_{AB} = \frac{\pi r_1}{2v_A}$$

$$t_{AB} = \frac{\pi(21\,\mathrm{m})}{2(40\,\mathrm{km/hr})\left(\frac{\mathrm{hr}}{3600\,\mathrm{s}}\right)\left(\frac{1000\,\mathrm{m}}{\mathrm{km}}\right)}$$

$$t_{AB} = 3.0\,\mathrm{s}$$

We can find the maximum constant speed at which the car can travel through the bend from C to D by letting $\|\vec{a}_{CD}\| = a_{CD,n} = a_{n,\max}$:

(1) $\Rightarrow$

$$a_{n,\max} = \frac{v_C^2}{r_2}$$

$$v_C = \sqrt{a_{n,\max} r_2}$$

$$v_C = \sqrt{0.75(9.81\,\mathrm{m/s^2})(24\,\mathrm{m})}$$

$$v_C = 13.3\,\mathrm{m/s} = 47.88\,\mathrm{km/hr}$$

Therefore, the time it takes to go from C to D is given by

(2) $\Rightarrow$

$$t_{CD} = \frac{\pi r_2}{2v_C}$$

$$t_{CD} = \frac{\pi(24\,\mathrm{m})}{2(13.3\,\mathrm{m/s})}$$

$$t_{CD} = 2.83\,\mathrm{s}$$

Now that we have the car's speed at C (and we already know that the speeds at A and B are the same), we can determine the constant acceleration in the straightaway:

(3) $\Rightarrow$

$$a_{BC} = \frac{v_C^2 - v_B^2}{2d}$$

$$a_{BC} = \frac{(13.3\,\mathrm{m/s})^2 - \left[(40\,\mathrm{km/hr})\left(\frac{\mathrm{hr}}{3600\,\mathrm{s}}\right)\left(\frac{1000\,\mathrm{m}}{\mathrm{km}}\right)\right]^2}{2(12\,\mathrm{m})}$$

$$\boxed{a_{BC} = 2.226\,\mathrm{m/s^2}}$$

The corresponding time of travel is

(4) $\Rightarrow$

$$t_{BC} = \frac{v_C - v_B}{a_{BC}}$$

$$t_{BC} = \frac{13.3\,\mathrm{m/s} - (40\,\mathrm{km/hr})\left(\frac{\mathrm{hr}}{3600\,\mathrm{s}}\right)\left(\frac{1000\,\mathrm{m}}{\mathrm{km}}\right)}{2.226\,\mathrm{m/s^2}}$$

$$t_{BC} = 0.98\,\mathrm{s}$$

Lastly, the total time it takes to go from A to D is

$$t_{\text{total}} = t_{AB} + t_{BC} + t_{CD}$$

$$t_{\text{total}} = 3.0\,\mathrm{s} + 2.83\,\mathrm{s} + 0.98\,\mathrm{m}$$

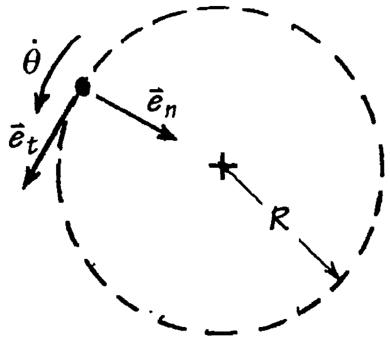
$$\boxed{t_{\text{total}} = 6.81\,\mathrm{s}}$$

2.4.19

GOAL: Determine $\|\vec{a}\|$ in g's, $\dot{\theta}$, $\ddot{\theta}$.

GIVEN: The end of the test tube in the centrifuge has tangential velocity $v_t = 3\,\mathrm{m/s}$ and tangential acceleration $a_t = 3\,\mathrm{m/s^2}$ at the given instant. The radius of the path is $R = 0.23\,\mathrm{m}$.

DRAW:



FORMULATE EQUATIONS:

The acceleration of the end of the test tube in terms of path coordinates is

$$\vec{a} = a_t \vec{e}_t + \frac{v_t^2}{R} \vec{e}_n \tag{1}$$

SOLVE:

(1) $\Rightarrow$

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_t^2 + \left(\frac{v_t^2}{R}\right)^2}$$

$$\|\vec{a}\| = \sqrt{(3\,\mathrm{m/s^2})^2 + \left(\frac{(3\,\mathrm{m/s})^2}{0.23\,\mathrm{m}}\right)^2}$$

$$\boxed{\|\vec{a}\| = 39.25\,\mathrm{m/s^2} = 4.000g}$$

(1) $\Rightarrow$

$$\frac{v_t^2}{R} = R\dot{\theta}^2$$

$$\begin{aligned}\dot{\theta} &= \frac{v_t}{R}\\ \dot{\theta} &= \frac{3\,\mathrm{m/s}}{0.23\,\mathrm{m}}\end{aligned}$$

$$\boxed{\dot{\theta} = 13.04\,\mathrm{rad/s}}$$

(1) $\Rightarrow$

$$a_t = R\ddot{\theta}$$

$$\begin{aligned}\ddot{\theta} &= \frac{a_t}{R}\\ \ddot{\theta} &= \frac{3\,\mathrm{m/s^2}}{0.23\,\mathrm{m}}\end{aligned}$$

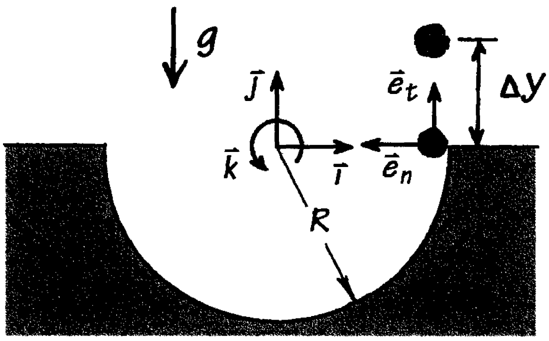
$$\boxed{\ddot{\theta} = 13.04\,\mathrm{rad/s^2}}$$

2.4.20

GOAL: Determine a_n and v_t before the snowboarder becomes airborne. Also find the maximum height achieved and the total time in the air.

GIVEN: Before becoming airborne, the snowboarder has $\|\vec{a}\| = 6\,\text{m/s}^2$ and $a_t = 3\,\text{m/s}^2$. The halfpipe has a radius $R = 6\,\text{m}$.

DRAW:



FORMULATE EQUATIONS:

The acceleration of the snowboarder at the edge of the halfpipe in terms of path coordinates is

$$\vec{a} = a_t\,\vec{e}_t + a_n\,\vec{e}_n = a_t\,\vec{e}_t + \frac{v_t^2}{R}\vec{e}_n \tag{1}$$

The maximum height achieved is governed by

$$v_t^2 = 2g\Delta y \tag{2}$$

The time to reach the maximum height is governed by

$$v_t = g\Delta t \tag{3}$$

SOLVE:

(1) $\Rightarrow$

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_t^2 + a_n^2}$$

$$\begin{aligned} a_n &= \sqrt{\|\vec{a}\|^2 - a_t^2} \\ a_n &= \sqrt{(6\,\text{m/s}^2)^2 - (3\,\text{m/s}^2)^2} \end{aligned}$$

$$\boxed{a_n = 5.2\,\mathrm{m/s^2}} \tag{4}$$

$(1) \Rightarrow$

$$a_n = \frac{v_t^2}{R}$$
$$v_t = \sqrt{a_n R} \tag{5}$$

$(4) \rightarrow (5) \Rightarrow$

$$v_t = \sqrt{(5.2\,\mathrm{m/s^2})(6\,\mathrm{m})}$$

$$\boxed{v_t = 5.59\,\mathrm{m/s}} \tag{6}$$

$(2) \Rightarrow$

$$\Delta y = \frac{v_t^2}{2g} \tag{7}$$

$(6) \rightarrow (7) \Rightarrow$

$$\Delta y = \frac{(5.59\,\mathrm{m/s})^2}{2(9.81\,\mathrm{m/s^2})}$$

$$\boxed{\Delta y = 1.59\,\mathrm{m}}$$

$(3) \Rightarrow$

$$\Delta t = \frac{v_t}{g} \tag{8}$$

$(6) \rightarrow (8) \Rightarrow$

$$\Delta t = \frac{5.59\,\mathrm{m/s}}{9.81\,\mathrm{m/s^2}}$$
$$\Delta t = 0.57\,\mathrm{s}$$

The total time in the air is simply twice the time to the maximum height, so

$$\begin{aligned} t_{\mathrm{total}} &= 2\Delta t \\ t_{\mathrm{total}} &= 2(0.57\,\mathrm{s}) \end{aligned}$$

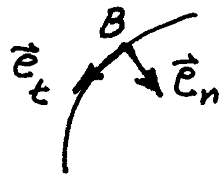
$$\boxed{t_{\mathrm{total}} = 1.14\,\mathrm{s}}$$

2.4.22

GOAL: Time before bicyclist B slips.

GIVEN: Initial conditions, dimensions, maximum sustainable acceleration, constraint of constant speed.

DRAW:



FORMULATE EQUATIONS:

Acceleration of B
$$\vec{a}_B = a_t \vec{e}_t + \frac{v^2}{r_C} \vec{e}_n \tag{1}$$

SOLVE:
$$32\,\text{km/hr} = 8.9\,\text{m/s}$$

(1) and constant speed condition$\Rightarrow$
$$\vec{a}_B = \frac{v^2}{r_C} \vec{e}_n = 8.5\,\text{m/s}^2 \vec{e}_n \tag{2}$$

(2) $\Rightarrow$
$$r_C = \frac{(8.9\,\text{m/s})^2}{8.5\,\text{m/s}^2} = 9.32\,\text{m} \tag{3}$$

(3) $\Rightarrow$
$$r_C = 15 - 8.2 \times 10^{-3} s^2 = 9.32\,\text{m} \Rightarrow s = 26.32\,\text{m}$$

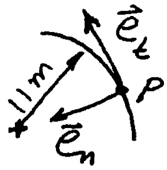
$$\boxed{t = \frac{26.32\,\text{m}}{8.9\,\text{m/s}} = 3\,\text{s}}$$

2.4.23

GOAL: Find $|v_P|$ and $|a_P|$ of a point P on a ferris wheel.

GIVEN: Wheel's radius, $\dot{\theta}$ as a function of time and elapsed time.

DRAW



FORMULATE EQUATIONS:

$$\vec{v}_P = v\,\vec{e}_t \tag{1}$$

$$\vec{a}_P = \dot{v}\,\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n \tag{2}$$

SOLVE:

$$\dot{\theta} = ct \Rightarrow \ddot{\theta} = c \tag{3}$$

Because P is moving in a circular fashion,

$$v = r_C\dot{\theta} = (11\text{ m})\dot{\theta} \tag{4}$$

(3), (4) $\rightarrow$ (1) $\Rightarrow$ $\quad \vec{v}_P = (11\text{ m})[(0.05\text{ rad/s}^2)(5\text{ s})] = 2.75\text{ m/s}$

$$\boxed{v = |v_p| = 2.75\text{ m/s}}$$

(3), (4) $\Rightarrow$ $$\dot{v} = r_C\ddot{\theta} = (11\text{ m})(0.05\text{ rad/s}^2) = 0.55\text{ m/s}^2 \tag{5}$$

(4), (5) $\rightarrow$ (2) $\Rightarrow$ $\quad \vec{a}_P = (0.55\text{ m/s}^2)\vec{e}_t + \frac{(2.75\text{ m/s})^2}{11\text{ m}}\vec{e}_n = (0.55\vec{e}_t + 0.69\vec{e}_n)\text{ m/s}^2$

$$\boxed{|\vec{a}_P| = 0.88\text{ m/s}^2}$$

2.4.24

GOAL: Find a track's radius of curvature.

DRAW:

FORMULATE EQUATIONS:

To solve this problem, start with the equation $ads = vdv$. The acceleration is constant, $a = 1.5\,\mathrm{m/s^2}$. So, integrating the equation

$$\int_{s_0}^{s_f} ads = \int_{v_0}^{v_f} vdv$$

gives

$$a(s_f - s_0) = \frac{1}{2}(v_f^2 - v_0^2)$$

SOLVE:

Using the given values gives

$$(1.5\ \mathrm{m/s^2})(180\ \mathrm{m}) = \frac{1}{2}(v_f^2 - (24\ \mathrm{m/s})^2)$$

So the value for the final velocity is

$$v_f = 33.4\,\mathrm{m/s}$$

We know the magnitude of the car's total acceleration is $4.45\,\mathrm{m/s^2}$ at C. The equation for the magnitude of the total acceleration is

$$||\vec{a}|| = \sqrt{a_n^2 + a_t^2}$$

Since $a_n = \dfrac{v^2}{r_C}$ we can compute the radius of curvature

$$4.45\ \mathrm{m/s^2} = \sqrt{\left(\frac{(33.4\ \mathrm{m/s})^2}{r_C}\right)^2 + (1.5\ \mathrm{m/s^2})^2}$$

and

$$\boxed{r_C = 266.3\,\mathrm{m}}$$

2.4.30

GOAL: Determine whether or not a car will lose traction during a turn.

GIVEN: The radius of curvature at entry and exit of the turn as well as the car's speed.

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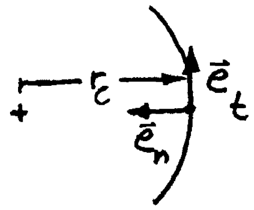


FORMULATE EQUATIONS:

All we need is the path coordinate form for acceleration:

$$\vec{a}_c = \dot{v}\,\vec{e}_t + \frac{v^2}{\rho_c}\vec{e}_n$$

SOLVE:

In this problem $\dot{v} = 0$ so

$$\vec{a}_c = \frac{v^2}{\rho_c}\vec{e}_n$$

Converting from km/h to m/s we have $161\,\text{km/hr} = 44.72\,\text{m/s}$

At the start of the curve we have

$$\frac{v^2}{r_c} = \frac{(44.72\ \text{m/s})^2}{305\ \text{m}} = 6.56\ \text{m/s}^2$$

At the end we have

$$\frac{v^2}{r_c} = \frac{(44.72)^2}{152.5} = 13.11\ \text{m/s}^2$$

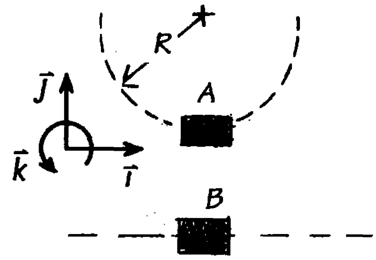
$13.11\ \text{m/s}^2$ exceeds the tire's maximum sustainable acceleration of $11\ \text{m/s}^2$ and so the car slips before exiting the turn.

2.5.1

GOAL: Determine $\vec{v}_{A/B}$ and $\vec{a}_{A/B}$.

GIVEN: Jet A has velocity $\vec{v}_A = 201\vec{i}$ m/s and acceleration $\vec{a}_{t,A} = 15\vec{i}$ m/s^2, and it is at the bottom of a loop with radius $R = 30\,\text{m}$. Jet B has velocity $\vec{v}_B = -213\vec{i}$ m/s and acceleration $\vec{a}_B = -12\vec{i}$ m/s^2.

DRAW:



FORMULATE EQUATIONS:

The velocity of jet A as seen by jet B is

$$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B \tag{1}$$

The acceleration of jet A as seen by jet B is

$$\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B = a_{t,A}\vec{i} + \frac{v_A^2}{R}\vec{j} - \vec{a}_B \tag{2}$$

SOLVE:

(1) $\Rightarrow$

$$\vec{v}_{A/B} = (201\,\text{m/s})\vec{i} - (-213\,\text{m/s})\vec{i}$$

$$\boxed{\vec{v}_{A/B} = 414\vec{i}\ \text{m/s}}$$

(2) $\Rightarrow$

$$\vec{a}_{A/B} = (15\,\text{m/s}^2)\vec{i} + \frac{(201\,\text{m/s})^2}{30\,\text{m}}\vec{j} - (-12\,\text{m/s}^2)\vec{i}$$

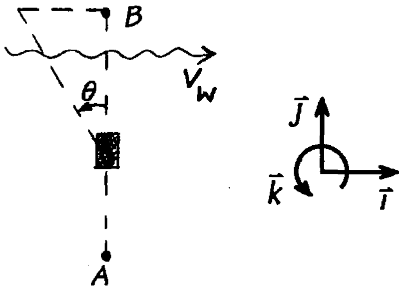
$$\boxed{\vec{a}_{A/B} = 27\vec{i} + 1346.7\vec{j}\ \text{m/s}^2}$$

2.5.2

GOAL: Determine how far over from B the plane will be at the anticipated time of arrival (i.e., with no wind) and at what speed and angle the plane should fly at relative to the wind to actually arrive at B in this time.

GIVEN: The plane initially has a velocity $\vec{v}_p = 322\vec{j}$ km/h and a strong wind blows at $\vec{v}_w = 129\vec{i}$ km/h when the plane is $d = 161\,\text{km}$ from B.

DRAW:



FORMULATE EQUATIONS:

The anticipated time of arrival t_B is based on when there is no wind, and so

$$t_B = \frac{d}{v_p} \tag{1}$$

The plane, if it doesn't course correct, will be pushed over a distance s to the right of B by the wind according to

$$s = v_w t_B \tag{2}$$

The actual velocity of the plane is given by

$$\vec{v}_{p,a} = \vec{v}_w + \vec{v}_{p/w} \tag{3}$$

SOLVE:

From (1), the plane's anticipated time of arrival is in

(1) ⇒
$$t_B = \frac{161\,\text{km}}{322\,\text{km/hr}} = 0.5\,\text{hr}$$

By (2), the plane will actually arrive to the right of B at a distance

(2) $\Rightarrow$
$$s = (129\,\text{km/h})(0.5\,\text{hr})$$
$$\boxed{s = 64.5\,\text{km}}$$

To arrive at B in the same amount of time as t_B, the actual velocity of the plane including the effects of the wind must be $\vec{v}_{p,a} = 322\vec{j}$ km/h, and so

$$\tan\theta = \frac{v_w}{v_{p,a}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{129\,\text{km/h}}{322\,\text{km/h}}\right)$$

$$\boxed{\theta = 21.8^\circ}$$

(3) $\Rightarrow$
$$v_{p,a}\vec{j} = v_w\vec{i} + v_{p/w}(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})$$

$\vec{i}$:
$$0 = v_w - v_{p/w}\sin\theta \quad \Rightarrow \quad v_{p/w} = \frac{129\,\text{km/h}}{\sin(21.8^\circ)}$$

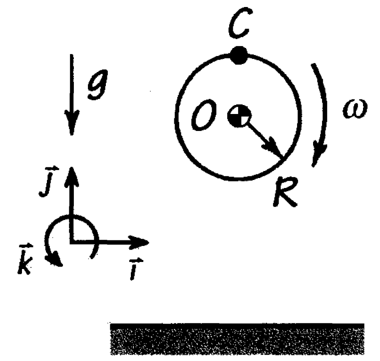
$$\boxed{v_{p/w} = 347\,\text{km/h}}$$

2.5.4

GOAL: Determine the speed and acceleration of the disk's top C.

GIVEN: The disk has a radius of $R = 18\,\text{cm}$ and spins at a constant angular speed of $\omega = 500\,\text{rpm}$ in the clockwise direction. The disk is released from rest far above the ground and allowed to fall $h = 0.9\,\text{m}$.

DRAW:



FORMULATE EQUATIONS:

The absolute velocity and acceleration of the disk's top C can be expressed as, respectively,

$$\vec{v}_C = \vec{v}_O + \vec{v}_{c/o} \tag{1}$$

$$\vec{a}_C = \vec{a}_O + \vec{a}_{c/o} \tag{2}$$

SOLVE:

Since the disk is released from rest, the speed of its mass center O after falling a distance h is given by $v_O = \sqrt{2gh}$, and thus the speed of the disk's top C is

(1) $\Rightarrow$

$$\vec{v}_C = -\sqrt{2gh}\,\vec{j} \; + R\omega\,\vec{i}$$

$$\|\vec{v}_C\| = \sqrt{\vec{v}_C \cdot \vec{v}_C} = \sqrt{(R\omega)^2 + 2gh}$$

$$\|\vec{v}_C\| = \sqrt{\left[\left(\frac{18\,\text{cm} \times 1\,\text{m}}{100\,\text{cm}}\right)\left(500\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right)\right]^2 + 2(9.81\,\text{m/s}^2)(0.9\,\text{m})}$$

$$\boxed{\|\vec{v}_C\| = 10.315\,\text{m/s}}$$

Noting that the disk is spinning at a constant angular speed and that its mass center is acted on by gravity alone, we have that the acceleration of the disk's top is

$$(2) \Rightarrow \qquad \vec{a}_C = -g\vec{j} - R\omega^2\vec{j}$$

$$\|\vec{a}_C\| = \sqrt{\vec{a}_C \cdot \vec{a}_C} = g + R\omega^2$$

$$\|\vec{a}_C\| = 9.81\,\mathrm{m/s^2} + (0.18\,\mathrm{m})\left[\left(500\frac{\mathrm{rev}}{\mathrm{min}}\right)\left(\frac{\mathrm{min}}{60\mathrm{s}}\right)\left(\frac{2\pi\,\mathrm{rad}}{\mathrm{rev}}\right)\right]^2$$

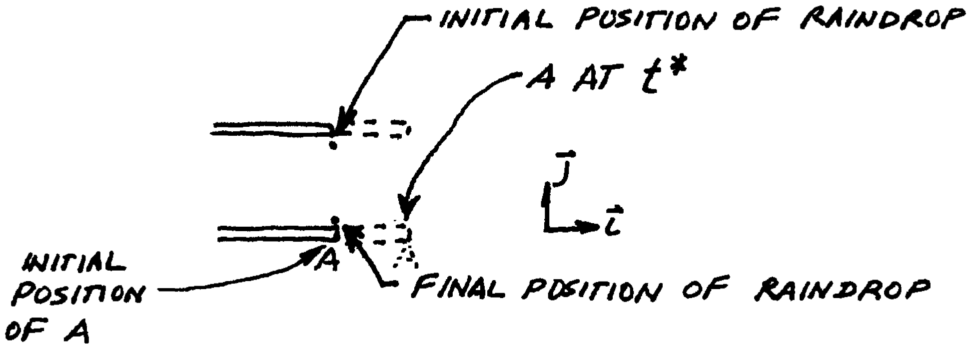
$$\boxed{\|\vec{a}_C\| = 502.79\,\mathrm{m/s^2}}$$

2.5.10

GOAL: Calculate the necessary constant acceleration for a raindrop to hit the floor of the train car eight feet from the point A and calculate the angle made by its velocity vector with the floor.

GIVEN: The initial speed of the train is $v_A = 24\,\mathrm{m/s}$, the speed of the raindrop is $v_{rd} = 12\,\mathrm{m/s}$ and the height of the overhang is $h = 2.4\,\mathrm{m}$.

DRAW



FORMULATE EQUATIONS:

Distance the point A travels in time t^*:

$$d = v_A t^* + \frac{a_A t^{*2}}{2} \tag{1}$$

Distance the raindrop falls in time t^*:

$$h = v_{rd} t^* \tag{2}$$

Relative velocity of the raindrop:

$$\vec{v}_{rd} = \vec{v}_A + \vec{v}_{rd/A} \tag{3}$$

SOLVE: First we'll determine the time at which the raindrop strikes the floor of the train car.

(2) $\Rightarrow$
$$2.4\,\mathrm{m} = (12\,\mathrm{m/s})t^* \;\Rightarrow\; t^* = 0.2\,\mathrm{s}$$

We want to ensure that in 0.2 s the point A moves forward by $2.4\,\mathrm{m}$:

(1) $\Rightarrow$
$$2.4\,\mathrm{m} = (24\,\mathrm{m/s})(0.2\,\mathrm{s}) + \frac{a_A(0.2\,\mathrm{s})^2}{2}$$

$$\boxed{a_A = -120\,\mathrm{m/s^2}}$$

(Note that this deceleration is *very* nonrealistic for an actual train.)

(3) $\Rightarrow$ $$\vec{v}_{rd} = \vec{v}_A + \vec{v}_{rd/_A} \;\Rightarrow\; \vec{v}_{rd/_A} = \vec{v}_{rd} - \vec{v}_A$$

$$\vec{v}_{rd/_A} = -12\vec{j}\ \text{m/s} - 24\vec{i}\ \text{m/s} + (120\,\text{m/s}^2)(0.2\,\text{s})\vec{i}\ = -12\vec{j}\ \text{m/s}$$

The deceleration is such that at 0.2 s the horizontal speed of the raindrop with respect to the train has gone to zero and the velocity is purely vertical.

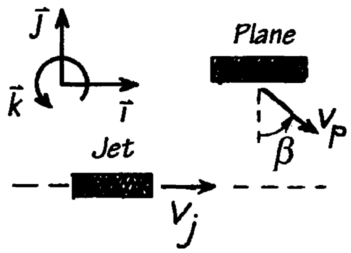
$$\boxed{\text{angle with respect to the floor is } 90^\circ}$$

2.5.11

GOAL: Determine the approach speed $\|\vec{v}_a\|$ of the fuel nozzle with respect to the jet, and find what speed the jet should slow down to if it wants to maintain the same $\|\vec{v}_a\|$ when the refueling plane reduces its speed.

GIVEN: The refueling plane comes in at an angle $\beta = 85°$ to the vertical with an initial speed $v_p = 563\,\text{km/h}$, while the jet flies at $\vec{v}_j = 580\vec{i}$ km/h. The plane then reduces its speed to $v_p^* = 547\,\text{km/hr}$.

DRAW:



FORMULATE EQUATIONS:

The approach velocity of the fuel nozzle as seen by the jet is

$$\vec{v}_a = \vec{v}_p - \vec{v}_j \tag{1}$$

SOLVE:

(1) $\Rightarrow$

$$\vec{v}_a = v_p(\sin\beta\,\vec{i} - \cos\beta\,\vec{j}\,) - v_j\,\vec{i}$$

$$\|\vec{v}_a\| = \sqrt{\vec{v}_a \cdot \vec{v}_a} = \sqrt{(v_p\sin\beta - v_j)^2 + (v_p\cos\beta)^2} \tag{2}$$

$$\|\vec{v}_a\| = \sqrt{\{(563\,\text{km/h})\sin(85°) - 580\,\text{km/h}\}^2 + \{(563\,\text{km/h})\cos(85°)\}^2}$$

$$\boxed{\|\vec{v}_a\| = 52.67\,\text{km/h}}$$

(2) $\Rightarrow$

$$\|\vec{v}_a\| = v_a = \sqrt{(v_p^*\sin\beta - v_j^*)^2 + (v_p^*\cos\beta)^2}$$

$$\begin{aligned} v_a^2 &= (v_p^*\sin\beta - v_j^*)^2 + (v_p^*\cos\beta)^2 \\ 0 &= (v_j^*)^2 - (2v_p^*\sin\beta)v_j^* + (v_p^*\sin\beta)^2 + (v_p^*\cos\beta)^2 - v_a^2 \\ 0 &= (v_j^*)^2 - (2v_p^*\sin\beta)v_j^* + \{(v_p^*)^2 - v_a^2\} \end{aligned}$$

$$
\begin{aligned}
v_j^* &= \frac{2v_p^* \sin\beta \pm \sqrt{(2v_p^* \sin\beta)^2 - 4\{(v_p^*)^2 - v_a^2\}}}{2} \\
v_j^* &= \frac{2(547\,\mathrm{km/h})\sin(85°) \pm \sqrt{\{2(547\,\mathrm{km/h})\sin(85°)\}^2 - 4\{(547\,\mathrm{km/h})^2 - (52.67\,\mathrm{km/h})^2\}}}{2} \\
v_j^* &= 522.53\,\mathrm{km/h},\; 567.31\,\mathrm{km/h}
\end{aligned}
$$

The slower speed corresponds to the jet and plane moving *away* from each other, so the jet speed we want is

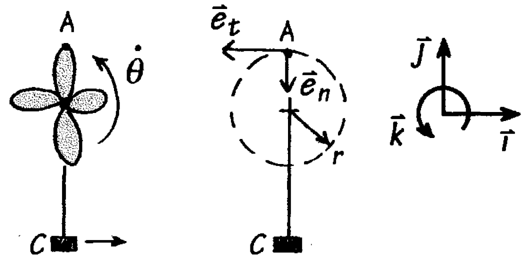
$$\boxed{v_j^* = 567.31\,\mathrm{km/h}}$$

2.5.12

GOAL: Determine $\|\vec{v}_A\|$ and $\|\vec{a}_A\|$ at the given instant.

GIVEN: The car travels with $\vec{v}_c = 48\vec{\imath}$ km/h and $\vec{a}_c = 1.5\vec{\imath}$ m/s^2. The windmill has a radius of $r = 10$ cm and spins at $\dot{\theta} = 200$rpm counterclockwise. At the given instant, $\vec{e}_t = -\vec{\imath}$ and $\vec{e}_n = -\vec{\jmath}$ at A.

DRAW:



FORMULATE EQUATIONS:

The velocity of point A on the windmill is given by

$$\vec{v}_A = \vec{v}_c + \vec{v}_{A/c} \tag{1}$$

The acceleration of A is given by

$$\vec{a}_A = \vec{a}_c + \vec{a}_{A/c} \tag{2}$$

SOLVE:

(1) $\Rightarrow$

$$\vec{v}_A = v_c\vec{\imath} + r\dot{\theta}\vec{e}_t = (v_c - r\dot{\theta})\vec{\imath}$$
$$\|\vec{v}_A\| = \sqrt{\vec{v}_A \cdot \vec{v}_A} = v_c - r\dot{\theta}$$
$$\|\vec{v}_A\| = 48\,\text{km/h} - 10\,\text{cm}\left(\frac{\text{m}}{100\,\text{cm}}\right)\left(\frac{\text{km}}{1000\,\text{m}}\right)(200\text{rpm})\left(\frac{60\text{min}}{\text{hr}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right)$$

$$\boxed{\|\vec{v}_A\| = 40.5\,\text{km/h}}$$

(2) $\Rightarrow$

$$\vec{a}_A = a_c\vec{\imath} + r\dot{\theta}^2\vec{e}_n = a_c\vec{\imath} - r\dot{\theta}^2\vec{\jmath}$$
$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \sqrt{a_c^2 + (r\dot{\theta}^2)^2}$$
$$\|\vec{a}_A\| = \sqrt{(1.5\,\text{m/s}^2)^2 + \left[\left(\frac{10}{100}\,\text{m}\right)\left(\frac{200\times 2\pi}{60}\,\text{rad/s}\right)^2\right]^2}$$

$$\boxed{\|\vec{a}_A\| = 43.9\,\text{m/s}^2}$$

2.5.13

GOAL: Find the agent's acceleration so that he will be moving at zero velocity with respect to the ground. Also, find the point on the ground where he will land.

GIVEN: Agent is 9 m from the back of the bus and moving with a constant acceleration with respect to the bus. Bus is moving at 16 km/h with respect to ground.

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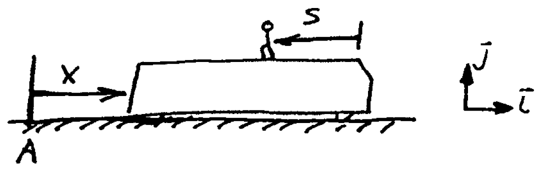


FORMULATE EQUATIONS:

To move at zero velocity with respect to the ground the agent must be moving at -16 km/h with respect to the bus. You can solve for the acceleration with respect to the bus using

$$\ddot{s} ds = \dot{s}\, d\dot{s} \tag{1}$$

Converting the speed from km/h to m/sec gives

$$16\,\text{km/h} = 4.4\,\text{m/sec}$$

SOLVE: Integrating (1) yields

$$\int_0^9 \ddot{s}\, ds = \int_0^{4.4} \dot{s}\, d\dot{s}$$

$$\ddot{s}(9) = \frac{1}{2}\left(4.4^{\,2}\right) \Rightarrow \ddot{s} = 1.08 \text{ m/s}^2$$

When the agent leaves the bus, he is traveling with zero velocity with respect to the ground in the $\vec{\imath}$ direction and thus he'll drop straight down from the back of the bus. To determine the location of the bus when the agent jumps, we look at the equation for change of position under constant acceleration

$$s = \frac{1}{2}\ddot{s}t^2 + \dot{s}_0 t + s_0$$

where $\dot{s}_0$ and s_0 are the speed and position of the agent with respect to the bus at $t = 0$. Since $\dot{s}_0$ and s_0 equal zero, the equation simplifies to

$$9 \text{ m} = \frac{1}{2}(1.08 \text{ m/s}^2)t^2$$

Solving for t gives $t = 4.08\,\text{s}$. Using this fact and the fact that the bus is moving at a constant speed gives us

$$x = \dot{x}t$$

$$x = (4.4 \text{ m/s})(4.08 \text{ s}) = 17.95 \text{ m}$$

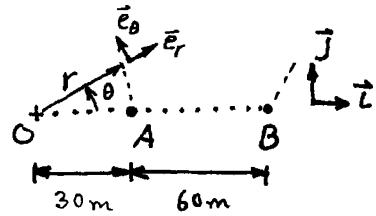
The position of the agent with respect to A is $\boxed{x = 17.95 \text{ m}}$

2.5.14

GOAL: Find the velocity and acceleration of car A relative to car B.

GIVEN: Absolute positions and velocities of cars A and B.

DRAW



FORMULATE EQUATIONS:

Velocity of car A relative to car B: $$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B \tag{1}$$

Acceleration of car A relative to car B: $$\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B \tag{2}$$

Absolute velocity of car A: $$\vec{v}_A = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{3}$$

Absolute acceleration of car A: $$\vec{a}_A = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \tag{4}$$

SOLVE:

(1) $\Rightarrow$
$$\vec{v}_{A/B} = \left[48\vec{j} - 96\left(0.5\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\right)\right] \text{ km/h} \tag{5}$$

$$\boxed{\vec{v}_{A/B} = \left(-48\vec{i} - 35.14\vec{j}\right) \text{ km/h}}$$

Because car A is moving at a constant speed, in a circle of constant radius, at angular position $\theta = 0$ we have

(3) $\Rightarrow$
$$\vec{v}_A = r\dot{\theta}\vec{e}_\theta = r\dot{\theta}\vec{j} \tag{6}$$

(4) $\Rightarrow$
$$\vec{a}_A = -r\dot{\theta}^2\vec{e}_r = -r\dot{\theta}^2\vec{i} \tag{7}$$

We can get $\dot{\theta}$ from (6) $\Rightarrow$ $\quad 48\,\text{km/h}\vec{j} = (30 \text{ m})\dot{\theta}\vec{j}$

$$\dot{\theta} = \left(\frac{48\,\text{km/h}}{30 \text{ m}}\right)\left(0.2778\frac{\text{m/s}}{\text{km/h}}\right) = 0.44 \text{ rad/s} \tag{8}$$

(8) $\rightarrow$ (7) $\Rightarrow$
$$\vec{a}_A = -(30 \text{ m})(0.44 \text{ rad/s})^2\vec{i} = -5.8 \text{ m/s}^2\vec{i} \tag{9}$$

Because car B is moving in a straight line: $$\vec{a}_B = 0 \tag{10}$$

(9), (10) $\rightarrow$ (2) $\Rightarrow$ $$\vec{a}_{A/B} = -5.8 \text{ m/s}^2\vec{i}$$

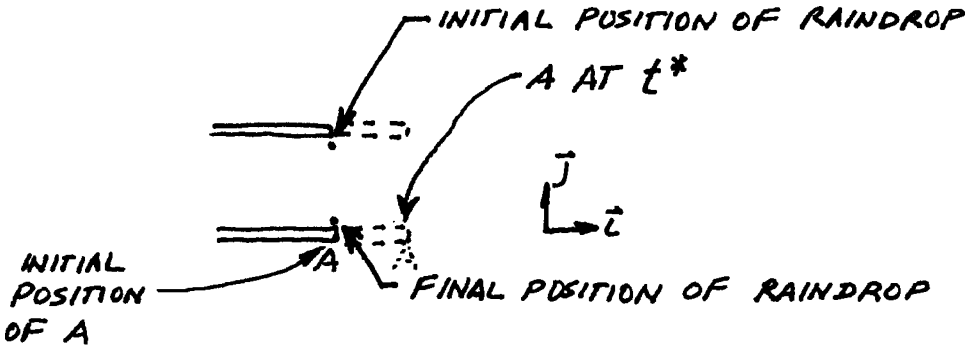
$$\boxed{\vec{a}_{A/B} = -5.8 \text{ m/s}^2\vec{i}}$$

2.5.16

GOAL: Find the distance raindrops travel into the train, and the velocity of raindrops relative to the train.

GIVEN: The speed of point A, v_A, the speed of the raindrops, v_{rd}, and the height of the overhang, h.

DRAW



FORMULATE EQUATIONS:

Distance the train travels in time t^*:

$$d = v_A t^* \tag{1}$$

Distance the raindrops fall in time t^*:

$$h = v_{rd} t^* \tag{2}$$

Relative velocity of the raindrop:

$$\vec{v}_{rd} = \vec{v}_A + \vec{v}_{rd_{/A}} \tag{3}$$

SOLVE:

A raindrop will fall by a height h in the same time as the train moves forward by a distance d.

(1), (2) $\Rightarrow$

$$d = \left(\frac{v_A}{v_{rd}}\right) h = \left(\frac{48\,\text{km/h}}{40\,\text{km/h}}\right)(2.1\,\text{m}) = 2.52\text{ m}$$

$$\boxed{d = 2.52\text{ m}}$$

Solving (3) for the velocity of the raindrop relative to point A gives us:

$$\vec{v}_{rd_{/A}} = \vec{v}_{rd} - \vec{v}_A = (-40\vec{j}\,)\,\text{km/h} - (48\vec{i}\,)\,\text{km/h}$$

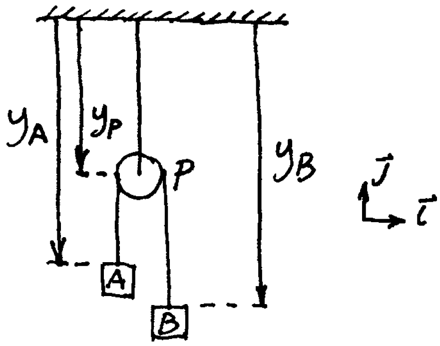
$$\boxed{\vec{v}_{rd_{/A}} = (-48\vec{i}\, - 40\vec{j}\,)\,\text{km/h}}$$

2.5.21

GOAL: Find the absolute velocity of A.

GIVEN: Pulley arrangement, speed of B and rate at which reel is pulling in rope.

DRAW



ASSUME: An overall constraint for this problem is the fact that the speed with which the pulley P drops is equal to the rate at which rope is fed from the reel.

FORMULATE EQUATIONS:

Ignoring motion of the pulley P we see by inspection that the velocities of A and B must be equal in magnitude and opposite in sign. We also have the relative velocity relationships

$$\vec{v}_A = \vec{v}_P + \vec{v}_{A/P}$$

$$\vec{v}_B = \vec{v}_P + \vec{v}_{B/P}$$

SOLVE:

We're given $\vec{v}_P = 36\vec{j}$ cm/s and $\vec{v}_B = -5\vec{j}$ cm/s. Using our relative velocity relationships gives us

$$\vec{v}_{B/P} = \vec{v}_B - \vec{v}_P$$

$$\vec{v}_{B/P} = -5\vec{j}\ \text{cm/s} - 36\vec{j}\ \text{cm/s} = -41\vec{j}\ \text{cm/s}$$

We've already observed that $\vec{v}_{A/P} = -\vec{v}_{B/P}$ and so have

$$\vec{v}_{A/P} = 41\vec{j}\ \text{cm/s}$$

We can now solve for $\vec{v}_A$.

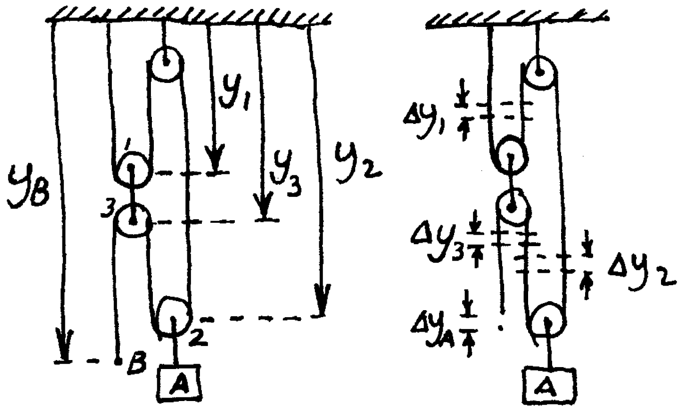
$$\vec{v}_A = \vec{v}_P + \vec{v}_{A/P} = 36\vec{j}\ \text{cm/s} + 41\vec{j}\ \text{cm/s} = \boxed{77\vec{j}\ \text{cm/s}}$$

2.5.23

GOAL: Find the absolute velocity of A.

GIVEN: Pulley arrangement and speed of the rope.

DRAW



FORMULATE EQUATIONS: In addition to the constraint that there is a fixed amount of rope, there is a constraint that the distance between pulley 1 and pulley 3 is fixed. This means that if you change the position of one, the position of the other will change by the same amount. Using the notation in the figure, this can be expressed as

$$\Delta y_3 = \Delta y_1 \tag{1}$$

Using the "conservation-of-rope" principle, the other constraint is

$$\Delta y_B - 2\Delta y_3 + 2\Delta y_2 + 2\Delta y_1 = 0 \tag{2}$$

SOLVE:

(1) → (2) ⇒
$$\Delta y_B + 2\Delta y_2 = 0 \tag{3}$$

Differentiating (3) and rearranging ⇒ $v_2 = -\frac{1}{2} v_B = -\frac{1.2\,\text{m/s}}{2} = -0.6\,\text{m/s}$

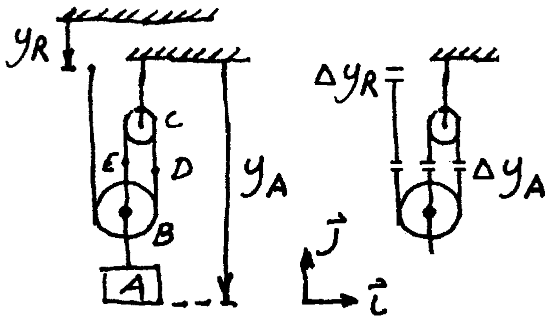
$$\boxed{\vec{v}_A = 0.6\vec{\jmath}\ \text{m/s}}$$

2.5.24

GOAL: Determine the velocity of a point on the rightmost rope of a pulley.

GIVEN: Pulley geometry and rate at which the reel is taking in pulley rope.

DRAW



ASSUME: To account for the reel we introduce a coordinate y_R that shows the vertical position of the leftmost pulley rope. A positive value for $\dot{y}_R$ indicates that the rope is unreeling and a negative value means the reel is retracting rope.

Conservation of rope gives us

$$-\Delta y_R + 3\Delta y_A = 0 \tag{1}$$

FORMULATE EQUATIONS:

(1) $\Rightarrow$

$$\dot{y}_A = \frac{1}{3}\dot{y}_R = \frac{25}{3}\text{ cm/s} \tag{2}$$

SOLVE:

We can proceed simply using observation at this point. We've already seen that the pulley B is moving up at $\frac{25}{3}$ cm/s. The rope that D's a part of goes up over the top pulley (C) and then down to pulley B. Hence, if B is rising at $\frac{25}{3}$ cm/s, the rope at D must be going down at the same rate.

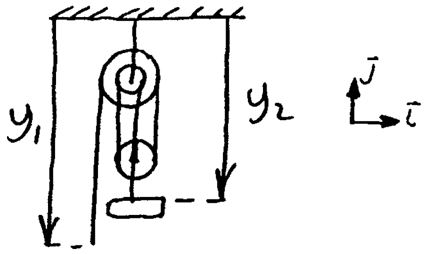
$$\boxed{\vec{v}_D = -\tfrac{25}{3}\vec{\jmath}\ \text{ cm/s}}$$

2.5.26

GOAL: Determine the acceleration of the mass given the acceleration of the free end of rope.

GIVEN: Free end of the rope is accelerating downward at 1.2 m/s^2.

DRAW:



FORMULATE EQUATIONS:

The displacement of the free end is Δy_1. The displacement of the mass is Δy_2. So the displacement relation is

$$3\Delta y_2 + \Delta y_1 = 0$$

Taking the limit as Δt goes to zero gives

$$\dot{y}_2 = -\frac{1}{3}\dot{y}_1$$

Differentiating again to find acceleration gives

$$\ddot{y}_2 = -\frac{1}{3}\ddot{y}_1$$

SOLVE:

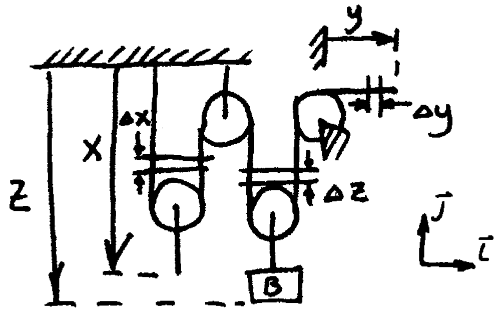
Letting $\ddot{y}_1 = 1.2\,\mathrm{m/s^2}$, we get $\boxed{\vec{a}_{mass} = 0.4\vec{j}\ \mathrm{m/s^2}}$.

2.5.28

GOAL: Find Block B's velocity.

GIVEN: Pulley geometry.

DRAW:



FORMULATE EQUATIONS:

$$\begin{aligned} 2\Delta x + 2\Delta z + \Delta y &= 0 \\ 2\dot{x} + 2\dot{z} + \dot{y} &= 0 \end{aligned} \tag{1}$$

SOLVE:

We're given $\dot{x} = 50$ cm/s and $\dot{y} = 25$ cm/s. Using this information in (1) yields

$$\dot{z} = \frac{-\dot{y} - 2\dot{x}}{2} = -\frac{1}{2}(25 \text{ cm/s} + 2(50 \text{ cm/s})) = -62.5 \text{ cm/s}$$

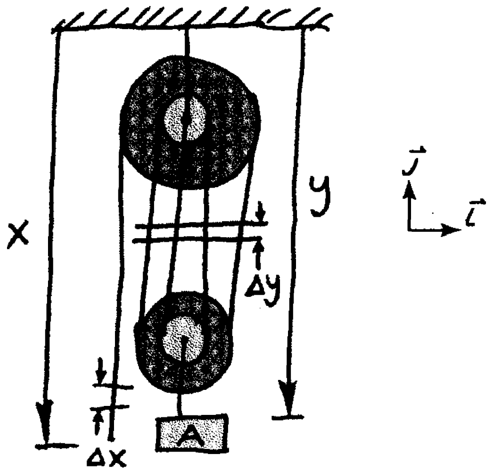
$$\boxed{\vec{v}_B = 62.5 \text{ cm/s}}$$

2.5.29

GOAL: Find $\vec{v}_A$ of a pulley system.

GIVEN: Pulley geometry.

DRAW:



ASSUME: The ropes connecting the pulleys are vertical.

FORMULATE EQUATIONS:

$$\Delta x + 4\Delta y = 0 \Rightarrow \ \dot{y} = -\frac{1}{4}\dot{x}$$

SOLVE:

We're given $\dot{x} = 25$ cm/s. Thus we have

$$\dot{y} = -\frac{1}{4}\dot{x} = -\frac{25}{4}\text{cm/s}$$

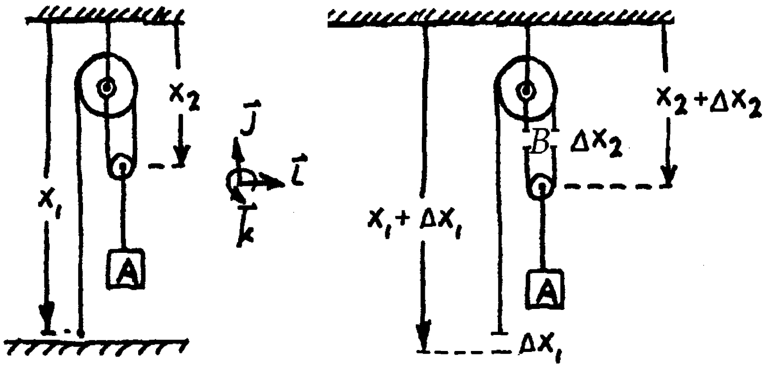
$$\boxed{\vec{v}_A = \tfrac{25}{4}\vec{\jmath} \ \ \text{cm/s}}$$

2.5.32

GOAL: Determine the velocity of the pulley rope's free end.

GIVEN: Block A is observed to be dropping down at a steady 0.27 m/s.

DRAW



FORMULATE EQUATIONS: We'll use the "conservation of rope" principle. To lower Block A we'll have to lower the pulley directly above it. To allow the pulley to drop we have to "break" the rope. If we increase x_2 by a small amount (to $x_2 + \Delta x_2$), then we'll have two gaps (shown in the figure at B) that will need to be filled in by additional rope. Similarly, bringing the free end of the pulley rope down means adding rope (bringing it from x_1 to $x_1 + \Delta x_1$). Thus, our conservation of rope condition is

$$\Delta x_1 + 2\Delta x_2 = 0 \tag{1}$$

Differentiating gives us

$$v_1 + 2\, v_2 = 0 \tag{2}$$

SOLVE:

(2) $\Rightarrow$

$$v_1 = -2v_2 = -2(0.27\,\text{m/s}) = -0.54\,\text{m/s}$$

x_1 is oriented downward and thus the negative value for v_1 indicates that the free end is moving upward and has a velocity of

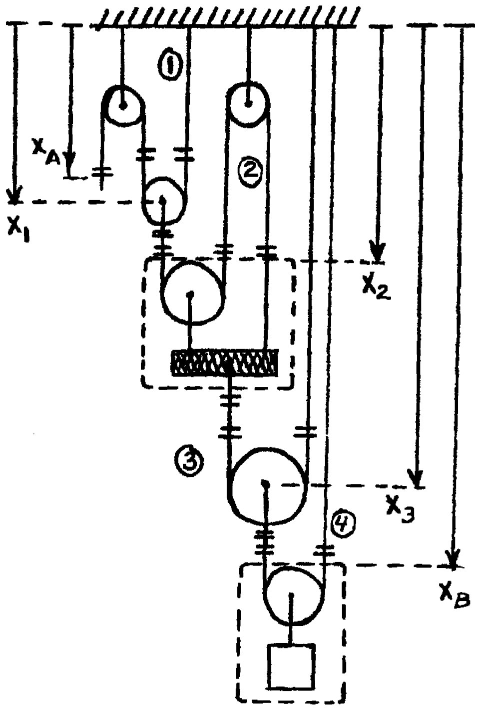
$$\boxed{\vec{v}_1 = 0.54\vec{\jmath}\ \text{ m/s}}$$

2.5.35

GOAL: Determine v_A.

GIVEN: Rope is reeled in at R at the rate $v_R = 0.9\,\text{m/s}$.

DRAW:



FORMULATE EQUATIONS:

By conservation of rope,

Rope 1 $\Rightarrow$

$$\Delta x_R + 2\Delta x_1 = 0 \tag{1}$$

Rope 2 $\Rightarrow$

$$-\Delta x_1 + 3\Delta x_A = 0 \tag{2}$$

SOLVE:

Differentiate (1) $\Rightarrow$

$$v_R + 2\dot{x}_1 = 0 \tag{3}$$

Differentiate (2) $\Rightarrow$

$$-\dot{x}_1 + 3v_A = 0 \tag{4}$$

(4) $\rightarrow$ (3) $\Rightarrow$

$$v_R + 6v_A = 0$$

$$
\begin{aligned}
v_A &= -\frac{1}{6}v_R \\
v_A &= -\frac{1}{6}(0.9\,\mathrm{m/s})
\end{aligned}
$$

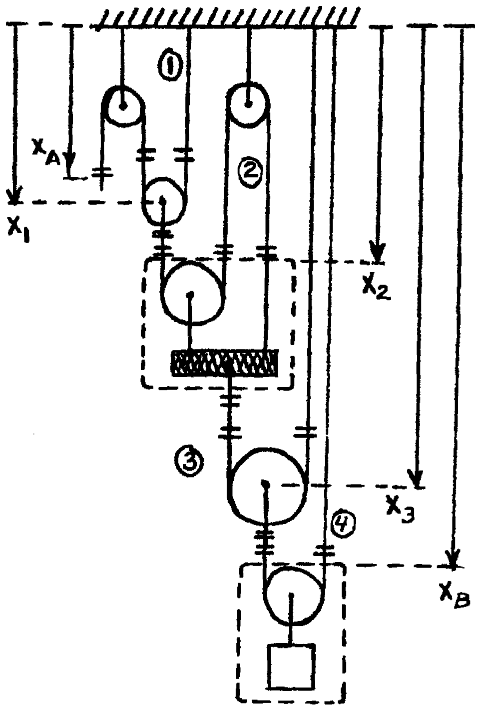
$$
\boxed{v_A = -0.15\,\mathrm{m/s}}
$$

2.5.36

GOAL: Determine v_B.

GIVEN: The person pulls down at A with a speed $v_A = 0.6\,\text{m/s}$.

DRAW:



FORMULATE EQUATIONS:

By conservation of rope,

Rope 1 ⇒
$$\Delta x_A + 2\Delta x_1 = 0 \tag{1}$$

Rope 2 ⇒
$$-\Delta x_1 + 3\Delta x_2 = 0 \tag{2}$$

Rope 3 ⇒
$$-\Delta x_2 + 2\Delta x_3 = 0 \tag{3}$$

Rope 4 ⇒
$$-\Delta x_3 + 2\Delta x_B = 0 \tag{4}$$

SOLVE:

Differentiate (1) ⇒
$$v_A + 2\dot{x}_1 = 0 \tag{5}$$

Differentiate (2) $\Rightarrow$
$$-\dot{x}_1 + 3\dot{x}_2 = 0 \tag{6}$$

Differentiate (3) $\Rightarrow$
$$-\dot{x}_2 + 2\dot{x}_3 = 0 \tag{7}$$

Differentiate (4) $\Rightarrow$
$$-\dot{x}_3 + 2v_B = 0 \tag{8}$$

(6), (7), (8) $\rightarrow$ (5) $\Rightarrow$
$$v_B = -\frac{1}{24}v_A$$
$$v_B = -\frac{1}{24}(0.6\,\mathrm{m/s})$$

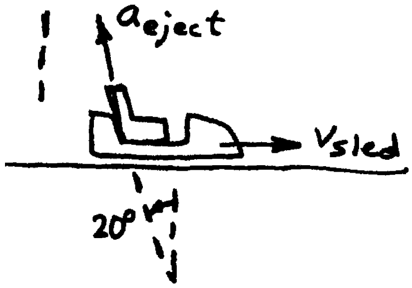
$$\boxed{v_B = -\tfrac{1}{40}\,\mathrm{m/s}}$$

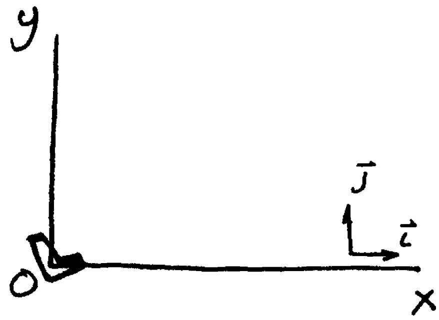
SA2.2

GOAL: Analyze the behavior of an ejector system.

GIVEN: System configuration and vehicle launch kinematics.

DRAW:





Ejection phase:

FORMULATE EQUATIONS: Motion in the x and y directions are independent. Gravity induces a negative vertical acceleration to the body and the ejection rocket provides an acceleration directed along the long axis of the seat. The seat ejects at an angle of 20° and so we have x and y components of the acceleration given by

$$a_x = -[16 - (9\,\mathrm{s}^{-2})t^2](\sin 20°)(9.81\,\mathrm{m})$$
$$a_y = [(16 - (9\,\mathrm{s}^{-2})t^2)\cos 20° - 1](9.81\,\mathrm{m})$$

SOLVE: The speed at launch is 600 KEAS= 600(1.85 km/h) = 308.3 m.

The acceleration components can be evaluated to give

$$a_x = -53.68\,\mathrm{m/s^2} + (30.2\,\mathrm{m/s^4})t^2$$
$$a_y = 137.7\,\mathrm{m/s^2} - (83\,\mathrm{m/s^4})t^2$$

Integration gives us

$$\begin{aligned} v_x &= -(53.68\,\mathrm{m/s^2})t + (10.07\,\mathrm{m/s^4})t^3 + 308.3\,\mathrm{m/s} \\ v_y &= (137.7\,\mathrm{m/s^2})t - (27.7\,\mathrm{m/s^4})t^3 \\ x &= -(26.84\,\mathrm{m/s^2})t^2 + (2.52\,\mathrm{m/s^4})t^4 + (308.3\,\mathrm{m/s})t \\ y &= (69\,\mathrm{m/s^2})t^2 - (6.93\,\mathrm{m/s^4})t^4 \end{aligned}$$

Evaluating at $t = 0.8\,\mathrm{s}$ yields

$$\begin{aligned} v_x &= 270.4\,\mathrm{m/s} \\ v_y &= 95.42\,\mathrm{m/s} \\ x &= \boxed{230.5\,\mathrm{m}} \\ y &= \boxed{41.32\,\mathrm{m}} \end{aligned}$$

Drag phase

FORMULATE EQUATIONS:

We're given that the acceleration follows

$$a_x = -0.003v_x^2$$

We can determine the change in distance as a function of the initial and final speed through evaluating

$$a\,dx = v\,dv \tag{1}$$

and find the elapsed time as a function of initial and final speed through

$$\frac{dv}{dt} = a \tag{2}$$

SOLVE: The drag phase ends when the seat reaches 100 KEAS= 100(1.85) km/h = 51.4 m/s

$$(1) \Rightarrow \quad \begin{aligned} -(0.01\,\mathrm{m}^{-1})v_x^2\,ds_x &= v_x\,dv_x \\ -(0.01\,\mathrm{m}^{-1})\,ds_x &= \frac{dv_x}{v_x} \\ -(0.01\,\mathrm{m}^{-1})s_x &= \ln[v_x]\Big|_f - \ln[v_x]\Big|_i \end{aligned}$$

Using $\ln[v_x]\Big|_f = \ln 51.4\,\mathrm{m/s}$ and $\ln[v_x]\Big|_i = \ln(270.4)\,\mathrm{m/s}$ gives us **[Q1]**

$$\Delta x = 166\,\mathrm{m}$$

$$(2) \Rightarrow \quad \begin{aligned} \frac{dv_x}{dt} &= -(0.01\,\mathrm{m}^{-1})v_x^2 \\ \frac{dv_x}{v_x^2} &= -(0.01\,\mathrm{m}^{-1})dt \\ -\left[\frac{1}{v_x}\Big|_f - \frac{1}{v_x}\Big|_i\right] &= -(0.01\,\mathrm{m}^{-1})\Delta t \end{aligned}$$

Solving gives us

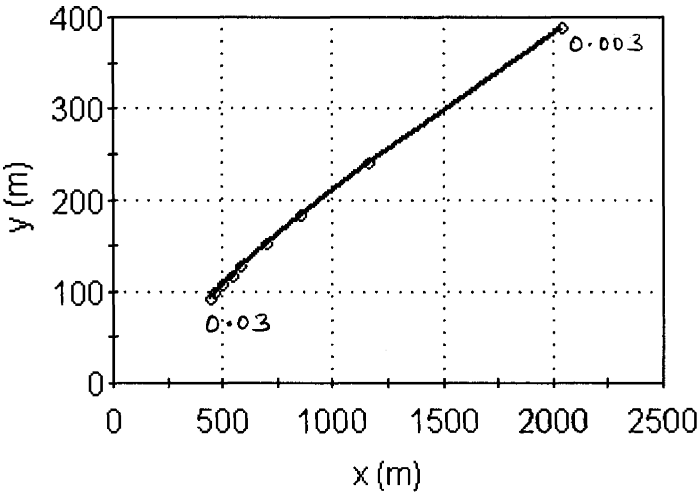
$$\Delta t = 1.58\,\mathrm{s}$$

Knowing how long the drag phase lasts and how far the seat travels due to the drag deceleration lets us calculate the final horizontal and vertical positions of the seat.

$$x = 230.5\,\mathrm{m} + (270.4\,\mathrm{m/s})(1.58) + 166\,\mathrm{ft} = \boxed{823.7\,\mathrm{m}}$$

$$y = 41.32\,\mathrm{m} + (95.42\,\mathrm{m/s})(1.58\,\mathrm{s}) - \frac{1}{2}(9.81\,\mathrm{m/s^2})(1.58\,\mathrm{s})^2 = \boxed{180\,\mathrm{m}}$$

Calculating the range variation for c going from $0.003\,\mathrm{m}^{-1}$ to $0.03\,\mathrm{m}^{-1}$ yields the following plot, which shows the x,y positions for increments of $0.03\,\mathrm{m}^{-1}$.


400
300
200
100
0
y (m)
0
500
1000
1500
2000
2500
x (m)
0·003
0·03

AUTHOR: Please verify this step. Logarithm is usually taken of dimensionless functions. Is it OK to take ln of a quantity having units (m/s in this case)?