Instructor's Solutions Manual
To accompany

Introduction to Linear Programming

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Chapter 6. Transportation Problems

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Preface

We give solutions or answers to all exercises in the textbook. Note that the exercises may have many correct solutions and even several correct answers. Some misprints in Answers to Selected Exercises (pages 305–317 in the textbook) are corrected in the solutions below.

Here are some other corrections.

Page 11. There are two Exercises 57. Both solved below.

Page 17. Replace Aarea in the table by Area.

Page 138. Replace dv in the second displayed line by $d = v$.

Page 165, Exercise 8. Replace $+-$ by $-$.

Page 210, Exercise 10. The empty entry means 0.

Page 240, Exercise 8. Replace p by p twice.

Page 240. There are two Exercises 12. Both solved below.

Page 249., line 6. Replace $p = 3$ by $p = 2$.

Page 250, Exercise 13. Replace $+1/\alpha^{t+1}$ by $-(-1/\alpha)^{t+1}$.

Chapter 1. Introduction

§1. What Is Linear Programming?

1. True.
2. True.
3. True.
4. True. $-35 \geq -36$.
5. True. This is because for real numbers any square and any absolute value are nonnegative.
6. False. Take $x = -1$.
7. False. For $x = -1$, $3(-1)^3 < 2(-1)^2$.
8. False (see Definition 1.5).
9. False (see Example 1.9 or 1.10).
10. False.
11. False. For example, the linear program

$$\text{Minimize } x + y \text{ subject to } x + y = 1$$

has infinitely many optimal solutions.

12. False.
13. True. It is a linear equation. A standard form is $4x = 8$ or $x = 2$.
14. Yes.
15. No. This is not a linear form, but an affine function.
16. Yes, if z is independent of x, y.
17. Yes if a and z do not depend on x, y.
18. No (see Definition 1.1).
19. No. But it is equivalent to a system of two linear constraints.
20. Yes, this is a linear equation.
21. Yes. We can write $0 = 0 \cdot x$, which is a linear form.
22. True if y is independent of x and hence can be considered as a given number; see Definition 1.3.
23. Yes if a, b are given numbers. In fact, this is a linear equation.
24. Yes, it is equivalent to the system of two linear constraints $-1 \leq x \leq 1$.
25. No. We will see later that any system of linear constrains gives a convex set. But we can rewrite the constraint as follows $x \geq 1$ OR $x \leq 1$. Notice the difference between OR and AND.
26. Yes, it is equivalent to $x = 0$.
27. See Problem 6.12.

28. We multiply the first equation by 5 and subtract the result from the second equation:

$$\begin{cases} x + 2y = 3 \\ \quad -y = -11. \end{cases}$$

Multiplying the second equation by -1, we solve it for y. Substituting this into the first equation, we find x. The answer is

$$\begin{cases} x = -19 \\ y = 11. \end{cases}$$

29. $x = 3 - 2y$ with an arbitrary y.

30. Multiplying the first equation by 3 and subtract the result from the second equation we obtain $0 = -9$ which can be simplified to $0 = 1$. So the system has no solutions.

31. $\min = 0$ at $x = y = 0, z = -1$. All optimal solutions are given as follows: $x = -y, y$ is arbitrary, $z = -1$.

32. $\min = 1$ when $-2 \le x \le -2.5$.

33. $\min = 1$ at $x = 0$.

34. The problem is unbounded.

35. $\min = 0$ at $x = -y = 1/2, z = -1$.

36. Yes, it is.

37. No. This is a linear equation.

38. No. Suppose $x + y^2 = ax + by$ with a, b independent of x, y. Setting $x = 0, y = 1$ we find that $b = 1$. Setting $x = 0, y = -1$ we find that $b = -1$.

39. No.

40. Yes, $y = 0 \cdot x + 1 \cdot y$.

41. Yes.

42. Yes, $(x+1)^2 + 2y - x^2 - 1 = 2x + 2y$.

43. No.

44. No. Suppose that xy is an affine function $ax + by + c$ of x, y. Setting $x = y = 0$, we find that $c = 0$. Setting $x = 0, y = 1$, we find that $b = 0$. Setting $x = 1, y = 0$, we find that $a = 0$. Setting $x = y = 1$, we find that $1 = 0$.

45. Yes.

46. No.

47. Yes.

48. No.

49. No.

50. No, this is a linear form.

51. Yes.

52. Yes, this is a linear equation because $0 = 0 \cdot x + 0 \cdot y$ is a linear form.

53. Yes. In fact, this is a linear equation.

54. Yes.

55. No. This is not even equivalent to any linear constraint with rational coefficients.

56. Yes.

57. No, see Exercise 44.

57. Let $f(x, y) = cx + dy$ be a linear form. Then $f(ax, ay) = cax + day = a(cx + dy) = af(x, y)$ for all a, x, y and $f(x_1 + x_2, y_1 + y_2) = c(x_1 + x_2) + d(y_1 + y_2) = cx_1 + dy_1 + cx_2 + dy_2 = f(x_1, y_1) + f(x_2, y_2)$ for all x_1, x_2, y_1, y_2.

58. We set $c = f(1, 0)$ and $d = f(0, 1)$. Then $f(x, y) = f(x + 0, 0 + y) = f(x, 0) + f(0, y) = f(1, 0)x + f(0, 1)y = cx + dy$ for all x, y.

59. $\min = 2^{-100}$ at $x = 0, y = 0, z = \pi/2, u = -100, v = -100$. In every optimal solution, x, y, u, v are as before and $z = \pi/2 + n\pi$ with any integer n such that $-32 \le n \le 31$. So there are exactly 64 optimal solutions.

§2. Examples of Linear Programs

1. Date:Nov 19, 2002. Store: http://www.peapod.com.

	Cereal by General Mills	Box Size	Price per oz	Price per box
A	APPLE CINNAMON CHEERIOS	15 OZ	\$.27	\$3.99
B	BASIC 4	16.2 OZ	\$.26	\$4.29
C	CHEERIOS	15 OZ	\$.25	\$3.79
F	FIBER ONE	16 OZ	\$.25	\$3.99
G	GOLDEN GRAHAMS	13 OZ	\$.31	\$3.99
H	HARMONY	16.7 OZ	\$.25	\$4.19
K	KIX	12.7 OZ	\$.28	\$3.59
L	LUCKY CHARMS	14 OZ	\$.30	\$4.19
M	Multi-Bran CHEX	15.6 OZ	\$.26	\$3.99
T	TOTAL Corn Flakes	10 OZ	\$.40	\$3.99

Food composition and Dietary Reference Intakes (DTIs) are taken from http://www.nal.usda.gov/fnic/etext/000020.html (Food and Nutrition Information Center, US Dept of Agriculture). The DRIs are actually a set of four reference values: Estimated Average Requirements (EAR), Recommended Dietary Allowances (RDA), Adequate Intakes (AI), and Tolerable Upper Intake Levels, (UL) that have replaced the 1989 Recommended Dietary Allowances (RDAs). DRIs for the vitamins are for a male of age 21. The RDA for protein is taken for 62.5 kg of body weight. Vitamin B_2 is also known as thiamin. Data are per serving unit. The DRIs represents daily requirements.

	serving size	Protein g	A IU	B_1 mg	C mg	B_6 mg	B_{12} mcg
A	30 g	1.8	500.1	0.375	6	0.501	1.5
B	55 g	4.3	392.7	0.297	0	0.39	1.155
C	30 g	3.3	500.1	0.375	6	0.501	1.5
F	30 g	2.4	0	0.375	6	0.501	1.5
G	30 g	1.5	500.1	0.375	6	0.501	1.5
H	100 g	11	909	2.73	55	1.82	7.6
K	30 g	1.8	529.2	0.375	6.3	0.501	1.5
L	30 g	2.1	500.1	0.375	6	0.501	1.5
M	49 g	3.43	445.41	0.333	5.39	0.446	1.323
T	30 g	1.815	428.7	1.5	60	2.001	6
DRI /	day	50	900	1.2	90	1.3	2.4

Our variables are amounts of foods. Since most of data are per serving size, we measure all variables in serving sizes. Now we compute prices per serving size. We use prices per box rather than less precise prices per ounce. We use the US standard: 1 pound = 453.59237 grams =16 ounces, hence 1 oz = 28.3495 g. We round off prices to 4 digits after the decimal point. For example, the price in dollars of one serving of APPLE CINNAMON CHEERIOS (A) is

$$30 \cdot 3.99/(15 \cdot 28.3495) = 0.281486 \approx 0.2815.$$

serial serial	variable (servings)	price per serving
A	a	\$ 0.2815
B	b	\$ 0.5138
C	c	\$ 0.2674
F	f	\$ 0.2640
G	g	\$ 0.3448
H	h	\$ 0.8850
K	k	\$ 0.2991
L	l	\$ 0.3167
M	m	\$ 0.4421
T	t	\$ 0.4222

Now we can write our objective function, the total cost (in dollars per day):

$$0.2815a + 0.5138b + 0.2674c + 0.2640f + 0.3448g + 0.8850h$$
$$+0.2991k + 0.3167l + 0.4421m + 0.4222t \to \min .$$

The constraints are

$a, b, c, f, g, h, k, l, m, t \geq 0,$

$$\begin{bmatrix} 1.8 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 4.3 & 392.7 & 0.297 & 0 & 0.39 & 1.155 \\ 3.3 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 2.4 & 0 & 0.375 & 6 & 0.501 & 1.5 \\ 1.5 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 11 & 909 & 2.73 & 55 & 1.82 & 7.6 \\ 1.8 & 529.2 & 0.375 & 6.3 & 0.501 & 1.5 \\ 2.1 & 500.1 & 0.375 & 6 & 0.501 & 1.5 \\ 3.43 & 445.41 & 0.333 & 5.39 & 0.446 & 1.323 \\ 1.815 & 428.7 & 1.5 & 60 & 2.001 & 6 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \\ f \\ g \\ h \\ k \\ l \\ m \\ t \end{bmatrix} \geq \begin{bmatrix} 50 \\ 900 \\ 1.2 \\ 90 \\ 1.3 \\ 2.4 \end{bmatrix}.$$

2. $\min = 1.525$ at $a = 0, b = 0.75, c = 0, d = 0.25$

3. $\max(P) = 100$ at $x_1 = 20, x_2 = 20,$

4. Let x be the number of quarters and y the number of dimes we pay. The program is

$$25x + 10y \to \min,$$

subject to

$$0 \leq x \leq 100,\ 0 \leq y \leq 90, 25x + 10y \geq C \text{ (in cents)}, x, y \text{ integers}.$$

This program is not linear because the conditions that x, y are integers. For $C = 15$, an optimal solution is $x = 0, y = 2$. For

$C = 102$, an optimal solution is $x = 3y = 3$ or $x = 1, y = 8$. For $C = 10000$, the optimization problem is infeasible.

5. Let x, y be the sides of the rectangle. Then the program is

$$xy \to \min,$$
subject to
$$x \geq 0,\ y \geq 0, 2x + 2y = 100.$$

Since $xy = x(50 - x) = 625 - (x - 25)^2 \leq 625$, max $= 625$ at $x = y = 25$.

6. If we choose the best worker for each work, we obtain the upper bound

$\max \leq 20 + 70 + 90 + 67$

for the objective function. However, this is not a matching, because B has two jobs, "a" and "d". If "a" is done by somebody else, the objective function drops by at least 5. If "d" is done by somebody else, the objective function drops by at least 12. So Ab,Bd,Cc,Da is the optimal matching with $\max = 15 + 70 + 90 + 67 = 242$.

7. We can compute the objective function at all 24 feasible solutions and find the following two optimal matchings: Ac, Ba, Cb, Dd and Ac, Bb, Ca, Dd with optimal value 7.

8. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9 + 9 + 7 + 9 + 9 = 43$ for the objective function. This bound cannot be achieved because of a conflict over c (the third column). So $\max \leq 42$. On the other hand, the matching aa, Bb, Cc, De, Ed achieved 42, so this is an optimal matching.

9. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9+9+9+9+8+9+6 = 59$ for the objective function. However looking at B and C, we see that they cannot get $9 + 9 = 18$ because of the conflict over g. They cannot get more than $7 + 9 = 16$. Hence, we have the upper bound max ≤ 57. On the other hand, we achieve this bound 57 in the matching Ac, Bf, Cg, De, Eb, Fd, Ga.

10. We find the best match for A, then the best remaining match for B, and so on. We obtain matching Aa, Bb, Cg, Df, Ec, Fd, Ge with $8 + 9 + 9 + 5 + 1 + 9 + 1 = 42$ (years) total. A similar procedure for columns yield the matching Aa, Bb, Cc, Fd, De, Ef, Gg with $8+9+7+9+1+5+4 = 43$ total, which is a better matching.

11. Let c_i be given numbers. Let c_j be an unknown maximal number (with unknown j). The linear program is

$$c_1x_1 + \cdots + c_nx_n \to \max, \text{ all } x_i \geq 0,\ x_1 + \cdots + x_n = 1.$$

Answer: $\max = c_j$ at $x_j = 1, x_i = 0$ for $i \neq j$.

12. $|a - x| + |b - x| + |c - x| \to \min$.

§3. Graphical Method

1. Let SSN be 123456789. Then the program is
$-x \to \max, 7x \leq 5, 13x \geq -8, 11x \leq 10$.
Answer: $\max = 8/13$ at $x = -8/13$.

2. Let SSN be 123456789. Then the program is $f = x - 3y \to \min, |6x + 4y| \leq 14, |5x + 7y| \leq 8, |x + y| \leq 17$.
Answer: $\min = -22$ at $x = -65/11, y = 59/11$.

3. Let SSN be 123456789. Then the program is
$x - 3y \to \min, |6x + 4y| \leq 14, |5x + 7y| \leq 8, |x + y| \leq 17$.
Answer: $\min = 242/11$ at $x = 65/11, y = -59/11$.

4. The first constrain is equivalent to 2 linear constraints $-7 \leq x \leq 3$. The feasable region for the second constraint is also an interval, $-8 \leq x \leq 2$. The feasable region for the linear program is the interval $-7 \leq x \leq 2$. In Case (i), the objective function is an increasing function of x and reaches its maximum 14 at the right endpoint $x = 2$. In Case (ii), the objective function is a decreasing function of x and reaches its maximum 63 at the left endpoint $x = -7$. In Case (ii),

$$\max = \begin{cases} 2b \text{ at } x = 2 & \text{if } b > 0, \\ 0 \text{ when } -7 \leq x \leq 2 & \text{if } b = 0, \\ -7b \text{ at } x = -7 & \text{if } b < 0. \end{cases}$$

5. $\max = 135$ at $x = -9, y = 18$

6. The objective function is not defined when $y = 0$. When $y = -1$ and $x \to \infty$, we have $x/y \to -\infty$. So this minimization problem is unbounded, $\min = -\infty$.

7. $\min = -1/4$ at $x = 1/2, y = -1/2$ or $x = -1/2, y = 1/2$.

8. $\max = 6$ at $x = 6, y = 0, z = 0$.

9. $\max = 1$ at $x = y = 0$

10. This is a linear equation for b. If $x \neq 0$, then the unique solution is $b = c/x$. If $x = 0 = c$, then every b is a solution. If $x = 0, c \neq 0$, then there are no solutions.

11. $\max = 22$ at $x = 4, y = 2$

12. Our variables, a and b are amounts of the foods in grams (including refuse). The constraints are
$a \geq 0, b \geq 0$,
$0.4 \cdot 5.78a + 0.98 \cdot 0.56b \geq 2000$ (energy in kcal),

$0.4 \cdot 0.0024a + 0.98 \cdot 0.0005b \geq 1.1(B_1 \text{ in mg}),$
$0.4 \cdot 0.0081a + 0.98 \cdot 0.0005b \geq 1.1(B_2 \text{ in mg}).$
The objective function in dollars to be minimized is

$2a/453.6 + 3b/0.4536.$

We took 1 pound $\approx$ 453.59 g (the exact value is 453.59237). The program can be solved graphically. However, it is clear that almond are both cheaper and have more contents for all 3 ingredients, so $b = 0$ in any optimal solution. Answer:
min $\approx$ \$5.05 at $a \approx 1146$ g ≈ 2.5 lb, $b = 0$.

13. The program is unbounded.

14. The feasible region van be given by 4 linear constraints: $-5 \leq x \leq 0, 2 \leq y \leq 3$. It is a rechtangle with 4 corners $[x, y] = [0, 3], [-5, 3][-5, -2], [0, -2]$. The objective function is not affine. Its level $|x| + y^2 = c$ is empty when $c < 0$, is a point when $c = 0$, and is made of 2 parabola pieces when $c > 0$. It is clear that max $= 14$ at $x = -5, y = 3$. The optimal solution is unique.

15. max $= 3$ at $x = y = 9, z = 1$. See the answer to Exercise 11 of §2.