Solutions for Selected Problems in Geometry: Theorems and Constructions

By
A. Berele and J. Goldman

Preface

To the instructor:

This manual contains solutions for half of the 200 problems in our geometry text. Our concentration has been primarily on problems in the first eight chapers, but later chapters are certainly represented. We were able to produce the prose portion of the problem solutions in Scientific Word (courtesy of Nydia Rodriguez - to whom we owe many thanks) and we hope they are succinct but readable. Because of time pressures, the figures are hand-drawn and not up to the graphics standards of the text itself. We hope to remedy this in any later editions. Primarily, we hope this manual is helpful to its users.

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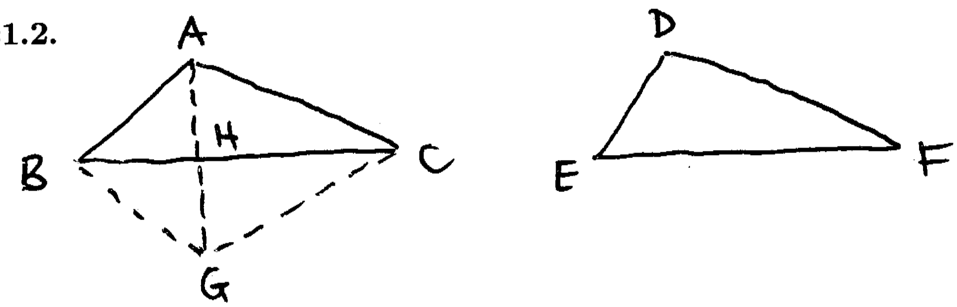
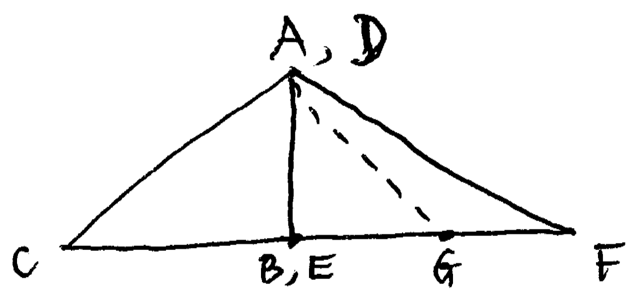
p. 17 #1.2.



Given $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$ in $\triangle ABC$ and $\triangle DEF$ respectively. We want to show that the two triangles are $\cong$. Construct G on the opposite side of $\overleftrightarrow{BC}$ from A so that $\angle GBC \cong \angle E$ and $\angle GCB \cong \angle F$. Work with the case where $\overline{AG}$ intersects $\overleftrightarrow{BC}$ in the point H, which is between B and C, as in the picture above. (The cases where $H = B, H = C$, H to the left of B, and H to the right of C are similar and must be done separately. We just do the one.) By ASA, we have $\triangle BGC \cong \triangle EDF$. From corresponding parts, $\overline{GC} \cong \overline{DF}$, and by transitivity, $\overline{AC} \cong \overline{GC}$. So, $\angle CAH \cong \angle CGH$. In like manner, obtain $\angle BAH \cong \angle BGH$. Add the angle measures to conclude $\angle BAC \cong \angle BGC$. Consequently, $\triangle BAC \cong \triangle BGC$, from SAS. Since $\cong$ is a transitive relation on $\triangle$, $\triangle BAC \cong \triangle EDF$ and we are finished.

p. 17, #1.3



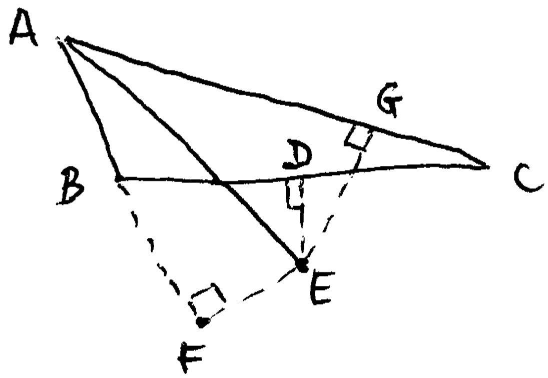
Assume the data of the problem. Because $\overline{AB} \cong \overline{DE}$, the triangles may be oriented as in the picture, A and D coincide, B and E coincide. $\angle ABC + \angle DEF = 180°$, so $\overleftrightarrow{CBF}$ is a line. From $\triangle ACF$ isosceles, obtain $\angle ACF \cong \angle AFC$. Claim: $\overline{CB} \cong \overline{BF}$; for, if not, one of CB or BF is smaller. Suppose $CB < BF$. Then there is a point G in the interior of $\overline{BF}$ such that $\overline{CB} \cong \overline{BG}$. But from SAS, $\triangle ABC \cong \triangle ABG$. $\therefore \angle AGC \cong \angle ACF \cong \angle AFC$. However, the exterior angle theorem implies $\angle AGC > \angle AFC$. Contradiction. Consequently $\overline{CB} \cong \overline{BF}$. $\therefore \triangle ABC \cong \triangle DEF$ by ASA.

p. 17, #1.4

Assume the data of the problem. Suppose $\overline{AB} \ncong \overline{DE}$. One of the last two segments is longer than the other, say $AB > DE$. Construct G on $\overline{BA}$ so that $\overline{BG} \cong \overline{ED}$. From SAS, $\triangle GBC \cong \triangle DEF$. $\therefore$ $\angle BGC \cong \angle D \cong \angle A$. However, from the exterior angle theorem, $\angle BGC > \angle A$. Contradiction: Thus we must have $\overline{AB} \cong \overline{DE}$ and therefore $\triangle ABC \cong \triangle DEF$ by ASA.

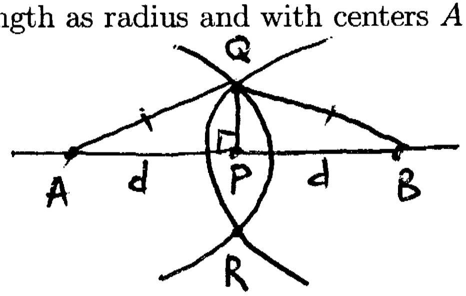
p. 17, #1.5

Consider the alternative diagram below. Note that E is not in the interior of $\triangle ABC$, that F is on $\overline{AB}$ extended, and that G is interior to $\overline{AC}$. As in the text, obtain $\overline{AF} \cong \overline{AG}$ and $\overline{BF} \cong \overline{GC}$; however, we must use subtraction to get $\overline{AB}$ and addition to get $\overline{AC}$. Thus, the "proof" will not work here. In fact, this is the situation in any scalene $\triangle$.



p. 17, #1.6

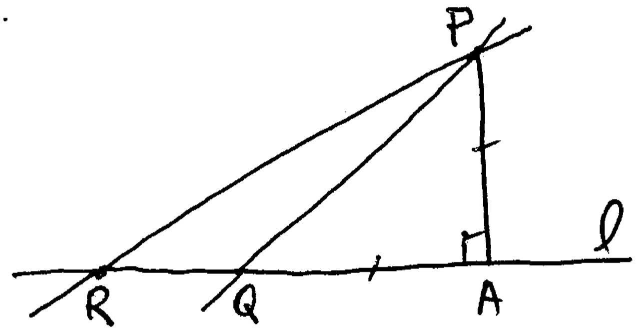
Set the points of your compass open to any convenient distance d and use it to determine points A and B on ℓ (by taking P as the center of a circle) such that $AP = PB = d$. Now, with compass points set on any convenient length apart $> d$, draw intersecting arcs of two circles, each with this length as radius and with centers A and B.



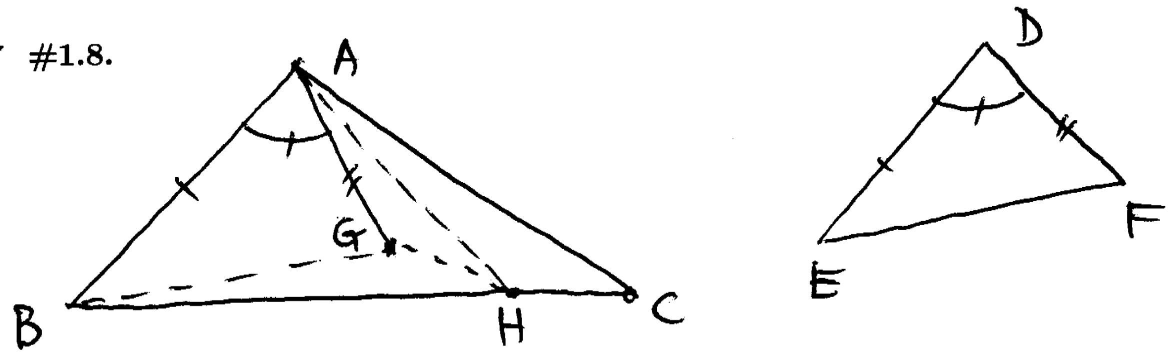
If Q is one of the intersection points of the two arcs, then one can show $\overleftrightarrow{PQ} \perp \ell$. The proof follows from the fact that $\triangle APQ \cong \triangle BPQ$ since we may use SSS. Further note that $\angle APQ = \angle BPQ$ and $\angle APQ + \angle BPQ = 180°$ imply that $\overleftrightarrow{PQ}$ meets ℓ at a $90°$ angle. Further congruence considerations yield the fact that the other intersection point, R, of the arcs lies on $\overleftrightarrow{PQ}$.

p. 17, #1.7

Drop a perpendicular $\overline{PA}$ to ℓ. Then lay-off $QA = PA$ with Q on ℓ. One can prove that $\angle PQA = 45°$ without using that the angle sum is $180°$ in a $\triangle$. It is easy to construct length $2 \cdot PA$. Use a compass to find R on $\overleftrightarrow{QA}$ such that $PR = 2 \cdot PA$. Here, it can be proved that $\angle PRA = 30°$.

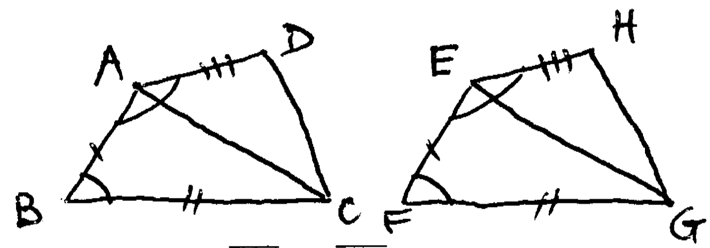


p. 17 #1.8.



From the information given, particularly that $\angle A > \angle D$, construct G, interior to $\angle BAC$ such that $\angle BAG \cong \angle D$ and $\overline{AG} \cong \overline{DF}$. The above picture illustrates the case when G is in the interior of $\triangle ABC$; the cases G on $\overline{BC}$ and G on the opposite side of $\overleftrightarrow{BC}$ from A must be considered separately, but won't be, here. By SAS, $\triangle ABG \cong \triangle DEF \Rightarrow BG = EF$. Construct the ray which bisects $\angle CAG$ and suppose it intersects $\overline{BC}$ at H. Draw $\overline{GH}$. From SAS and the definition of angle bisector, $\triangle AGH \cong \triangle ACH$, $\therefore HG = HC$. Also, in $\triangle BGH$, we have $BH + HG > BG = EF$. $\therefore BH + HC > EF$. $\therefore BC > EF$.

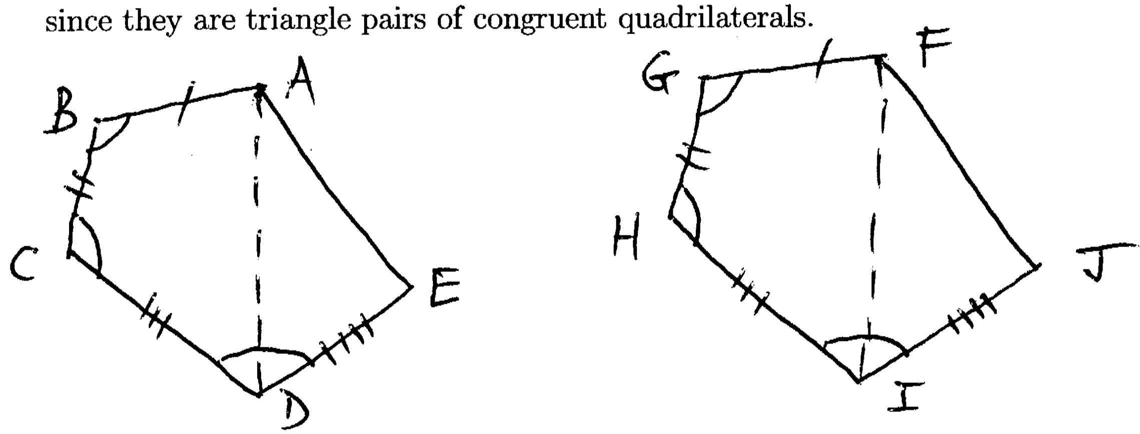
p. 17, #1.10



(a) Assume the data of part (a).
We must show 4 pairs of triangles formed by diagonalization are congruent.
We immediately have $\triangle ABC \cong \triangle EFG$ from SAS. Thus $\overline{AC} \cong \overline{EG}$, $\angle BAC \cong \angle FEG$, and $\angle BCA \cong \angle FGE$. By subtraction, $\angle DAC \cong \angle HEG$ and thus, by SAS, $\triangle DAC \cong \triangle HEG$. Thus, $\overline{DC} \cong \overline{HG}$, $\angle D \cong \angle H$, and $\angle DCA \cong \angle HGE$. By addition, $\angle C \cong LG$. We now have all the data we need to prove $\triangle ABD \cong \triangle EFH$ and that $\triangle BDC \cong \triangle FHG$. This shows $ABCD \cong EFGH$ for a convex figure. The proof is similar for a non-convex figure.

(b) Applications of SSS easily reduce this situation to the case of (a).

(c) Note that part (a) is an $SASAS$ theorem for quadrilaterals.
In the figures below, assume $\overline{AB} \cong \overline{FG}$, $\angle B = \angle G$, $\overline{BC} \cong \overline{GH}$, $\angle C = \angle H$, and $\overline{CD} \cong \overline{HI}$. This immediately gives $ABCD \cong FGHI$. In addition, complete the $SASASAS$ assumption, by assuming $\angle D \cong \angle I$ and $\overline{DE} \cong \overline{IJ}$. From the congruence of the quadrilaterals, $\overline{AD} \cong \overline{FI}$, and $\angle CDA = \angle HIF$. By subtraction $\angle ADE \cong \angle FIJ$, and by SAS, $\triangle ADE \cong \triangle FIJ$. Thus, $\angle J = \angle E$, $\overline{AE} \cong \overline{FJ}$, and, by addition, $\angle A = \angle F$. We now have all corresponding sides and angles congruent, consequently, pairs of triangles formed by subsets of any three corresponding vertices are congruent, since they are triangle pairs of congruent quadrilaterals.



(d) Draw a diagonal to produce a polygon of $n+1$ sides as the union of an n-sided polygon and a triangle (analogue to (c). Use the induction hypothesis on the polygon of n sides. Note the 2 triangles thus produced are congruent.

p. 27, #2.1

$a \Rightarrow b:$ Assume $\ell_1 || \ell_2$. $\therefore \angle a \cong \angle g$. But $\angle g$ and $\angle e$ are vertical angles. Thus $\angle a \cong \angle e$.

$b \Rightarrow c:$ given $\angle a \cong \angle e$. From vertical angle thm., $\angle a \cong \angle c$ and $\angle e \cong \angle g$ $\therefore \angle c \cong \angle g$.

$c \Rightarrow d:$ given $\angle c \cong \angle g$. But $\angle c \cong \angle a \Rightarrow \angle a \cong \angle g$. $\therefore 180° - \angle a \cong 180° - \angle g \therefore \angle b = 180° - \angle g = 180° - \angle e$, since $\angle g \cong \angle e$.

$d \Rightarrow e:$ given $\angle b \cong 180° - \angle e$. But $\angle d \cong \angle b \Rightarrow \angle d \cong 180° - \angle e \cong 180° - \angle g \cong \angle h$.

$e \Rightarrow a:$ given $\angle d \cong \angle h$. But $\angle d \cong \angle b \cong \angle h \Rightarrow \ell_1 || \ell_2$, by alternate interior angle $\Rightarrow$ parallel lines thm.