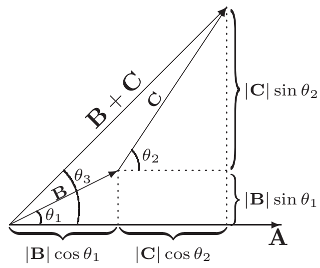
Chapter 1

Vector Analysis

Problem 1.1

(a) From the diagram, $|\mathbf{B}+\mathbf{C}|\cos\theta_3 = |\mathbf{B}|\cos\theta_1 + |\mathbf{C}|\cos\theta_2$. Multiply by $|\mathbf{A}|$.
$|\mathbf{A}||\mathbf{B}+\mathbf{C}|\cos\theta_3 = |\mathbf{A}||\mathbf{B}|\cos\theta_1 + |\mathbf{A}||\mathbf{C}|\cos\theta_2$.
So: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$. (Dot product is distributive)

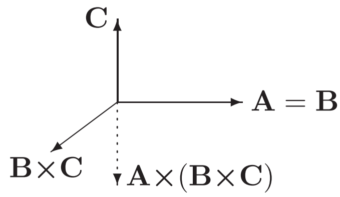
Similarly: $|\mathbf{B}+\mathbf{C}|\sin\theta_3 = |\mathbf{B}|\sin\theta_1 + |\mathbf{C}|\sin\theta_2$. Mulitply by $|\mathbf{A}|\,\hat{\mathbf{n}}$.
$|\mathbf{A}||\mathbf{B}+\mathbf{C}|\sin\theta_3\,\hat{\mathbf{n}} = |\mathbf{A}||\mathbf{B}|\sin\theta_1\,\hat{\mathbf{n}} + |\mathbf{A}||\mathbf{C}|\sin\theta_2\,\hat{\mathbf{n}}$.
If $\hat{\mathbf{n}}$ is the unit vector pointing out of the page, it follows that
$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C})$. (Cross product is distributive)



(b) For the general case, see G. E. Hay's *Vector and Tensor Analysis*, Chapter 1, Section 7 (dot product) and Section 8 (cross product)

Problem 1.2

The triple cross-product is *not* in general associative. For example, suppose $\mathbf{A} = \mathbf{B}$ and $\mathbf{C}$ is perpendicular to $\mathbf{A}$, as in the diagram. Then $(\mathbf{B}\times\mathbf{C})$ points out-of-the-page, and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ points *down*, and has magnitude ABC. But $(\mathbf{A}\times\mathbf{B}) = \mathbf{0}$, so $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{0} \neq \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

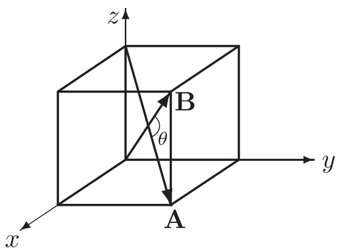


Problem 1.3

$\mathbf{A} = +1\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} - 1\,\hat{\mathbf{z}}$; $A = \sqrt{3}$; $\mathbf{B} = 1\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}$; $B = \sqrt{3}$.

$\mathbf{A}\cdot\mathbf{B} = +1+1-1 = 1 = AB\cos\theta = \sqrt{3}\sqrt{3}\cos\theta \Rightarrow \cos\theta = \frac{1}{3}$.

$$\boxed{\theta = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 70.5288^\circ}$$



Problem 1.4

The cross-product of any two vectors in the plane will give a vector perpendicular to the plane. For example, we might pick the base ($\mathbf{A}$) and the left side ($\mathbf{B}$):

$\mathbf{A} = -1\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}$; $\mathbf{B} = -1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}.$$

This has the right *direction*, but the wrong *magnitude*. To make a *unit* vector out of it, simply divide by its length:

$$|\mathbf{A}\times\mathbf{B}| = \sqrt{36+9+4} = 7. \qquad \hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \boxed{\tfrac{6}{7}\hat{\mathbf{x}} + \tfrac{3}{7}\hat{\mathbf{y}} + \tfrac{2}{7}\hat{\mathbf{z}}}.$$

Problem 1.5

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix}$$

$= \hat{\mathbf{x}}[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)] + \hat{\mathbf{y}}() + \hat{\mathbf{z}}()$

(I'll just check the x-component; the others go the same way)

$= \hat{\mathbf{x}}(A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z) + \hat{\mathbf{y}}() + \hat{\mathbf{z}}()$.

$\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = [B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)]\,\hat{\mathbf{x}} + ()\,\hat{\mathbf{y}} + ()\,\hat{\mathbf{z}}$

$= \hat{\mathbf{x}}(A_yB_xC_y + A_zB_xC_z - A_yB_yC_x - A_zB_zC_x) + \hat{\mathbf{y}}() + \hat{\mathbf{z}}()$. They agree.

Problem 1.6

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) + \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{A}(\mathbf{C}\cdot\mathbf{B}) + \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) - \mathbf{B}(\mathbf{C}\cdot\mathbf{A}) = \mathbf{0}$.

So: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) - (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.

If this is zero, then either $\mathbf{A}$ is parallel to $\mathbf{C}$ (including the case in which they point in *opposite* directions, or one is zero), or else $\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{A} = 0$, in which case $\mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{C}$ (including the case $\mathbf{B} = \mathbf{0}$.)

Conclusion: $\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{C} \iff \text{either } \mathbf{A} \text{ is parallel to } \mathbf{C}\text{, or } \mathbf{B} \text{ is perpendicular to } \mathbf{A} \text{ and } \mathbf{C}.}$

Problem 1.7

$$\boldsymbol{\mathscr{r}} = (4\,\hat{\mathbf{x}} + 6\,\hat{\mathbf{y}} + 8\,\hat{\mathbf{z}}) - (2\,\hat{\mathbf{x}} + 8\,\hat{\mathbf{y}} + 7\,\hat{\mathbf{z}}) = \boxed{2\,\hat{\mathbf{x}} - 2\,\hat{\mathbf{y}} + \hat{\mathbf{z}}}$$

$$\mathscr{r} = \sqrt{4+4+1} = \boxed{3}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \boxed{\tfrac{2}{3}\hat{\mathbf{x}} - \tfrac{2}{3}\hat{\mathbf{y}} + \tfrac{1}{3}\hat{\mathbf{z}}}$$

Problem 1.8

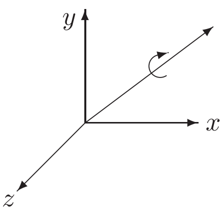
(a) $\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z = (\cos\phi A_y + \sin\phi A_z)(\cos\phi B_y + \sin\phi B_z) + (-\sin\phi A_y + \cos\phi A_z)(-\sin\phi B_y + \cos\phi B_z)$
$= \cos^2\phi A_yB_y + \sin\phi\cos\phi(A_yB_z + A_zB_y) + \sin^2\phi A_zB_z + \sin^2\phi A_yB_y - \sin\phi\cos\phi(A_yB_z + A_zB_y) + \cos^2\phi A_zB_z$
$= (\cos^2\phi + \sin^2\phi)A_yB_y + (\sin^2\phi + \cos^2\phi)A_zB_z = A_yB_y + A_zB_z.$ ✓

(b) $(\overline{A}_x)^2 + (\overline{A}_y)^2 + (\overline{A}_z)^2 = \Sigma_{i=1}^3 \overline{A}_i\overline{A}_i = \Sigma_{i=1}^3 \left(\Sigma_{j=1}^3 R_{ij}A_j\right)\left(\Sigma_{k=1}^3 R_{ik}A_k\right) = \Sigma_{j,k}\left(\Sigma_i R_{ij}R_{ik}\right)A_jA_k.$

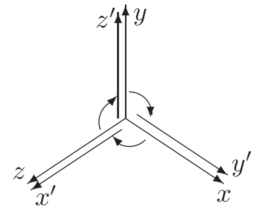
This equals $A_x^2 + A_y^2 + A_z^2$ *provided* $\boxed{\Sigma_{i=1}^3 R_{ij}R_{ik} = \left\{\begin{matrix} 1 \ \mathit{if}\ j = k \\ 0 \ \mathit{if}\ j \neq k \end{matrix}\right\}}$

Moreover, if R is to preserve lengths for *all* vectors $\mathbf{A}$, then this condition is not only *sufficient* but also *necessary*. For suppose $\mathbf{A} = (1,0,0)$. Then $\Sigma_{j,k}(\Sigma_i R_{ij}R_{ik})A_jA_k = \Sigma_i R_{i1}R_{i1}$, and this must equal 1 (since we want $\overline{A}_x^2 + \overline{A}_y^2 + \overline{A}_z^2 = 1$). Likewise, $\Sigma_{i=1}^3 R_{i2}R_{i2} = \Sigma_{i=1}^3 R_{i3}R_{i3} = 1$. To check the case $j \neq k$, choose $\mathbf{A} = (1,1,0)$. Then we want $2 = \Sigma_{j,k}(\Sigma_i R_{ij}R_{ik})A_jA_k = \Sigma_i R_{i1}R_{i1} + \Sigma_i R_{i2}R_{i2} + \Sigma_i R_{i1}R_{i2} + \Sigma_i R_{i2}R_{i1}$. But we already know that the first two sums are both 1; the third and fourth are *equal*, so $\Sigma_i R_{i1}R_{i2} = \Sigma_i R_{i2}R_{i1} = 0$, and so on for other unequal combinations of j, k. ✓ In matrix notation: $\tilde{R}R = 1$, where $\tilde{R}$ is the *transpose* of R.

Problem 1.9



Looking down the axis:



A 120° rotation carries the z axis into the y $(=\overline{z})$ axis, y into x $(=\overline{y})$, and x into z $(=\overline{x})$. So $\overline{A}_x = A_z$, $\overline{A}_y = A_x$, $\overline{A}_z = A_y$.

$$\boxed{R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

Problem 1.10

(a) $\boxed{\text{No change.}}$ $(\overline{A}_x = A_x,\ \overline{A}_y = A_y,\ \overline{A}_z = A_z)$

(b) $\boxed{\mathbf{A} \longrightarrow -\mathbf{A},}$ in the sense $(\overline{A}_x = -A_x,\ \overline{A}_y = -A_y,\ \overline{A}_z = -A_z)$

(c) $(\mathbf{A}\times\mathbf{B}) \longrightarrow (-\mathbf{A})\times(-\mathbf{B}) = (\mathbf{A}\times\mathbf{B})$. That is, if $\mathbf{C} = \mathbf{A}\times\mathbf{B}$, $\boxed{\mathbf{C} \longrightarrow \mathbf{C}}$. *No* minus sign, in contrast to behavior of an "ordinary" vector, as given by (b). If $\mathbf{A}$ and $\mathbf{B}$ are *pseudovectors*, then $(\mathbf{A}\times\mathbf{B}) \longrightarrow (\mathbf{A})\times(\mathbf{B}) = (\mathbf{A}\times\mathbf{B})$. So the cross-product of two pseudovectors is again a *pseudovector*. In the cross-product of a vector and a pseudovector, one changes sign, the other doesn't, and therefore the cross-product is itself a *vector*. *Angular momentum* $(\mathbf{L} = \mathbf{r}\times\mathbf{p})$ and *torque* $(\mathbf{N} = \mathbf{r}\times\mathbf{F})$ are pseudovectors.

(d) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \longrightarrow (-\mathbf{A})\cdot((-\mathbf{B})\times(-\mathbf{C})) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. So, if $a = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, then $\boxed{a \longrightarrow -a;}$ a pseudoscalar *changes sign* under inversion of coordinates.

Problem 1.11

$$\boxed{\begin{array}{l} (a)\nabla f = 2x\,\hat{\mathbf{x}} + 3y^2\,\hat{\mathbf{y}} + 4z^3\,\hat{\mathbf{z}} \\ (b)\nabla f = 2xy^3z^4\,\hat{\mathbf{x}} + 3x^2y^2z^4\,\hat{\mathbf{y}} + 4x^2y^3z^3\,\hat{\mathbf{z}} \\ (c)\nabla f = e^x \sin y \ln z\,\hat{\mathbf{x}} + e^x \cos y \ln z\,\hat{\mathbf{y}} + e^x \sin y (1/z)\,\hat{\mathbf{z}} \end{array}}$$

Problem 1.12

(a) $\nabla h = 10[(2y - 6x - 18)\,\hat{\mathbf{x}} + (2x - 8y + 28)\,\hat{\mathbf{y}}]$. $\nabla h = 0$ at summit, so

$$\left.\begin{array}{l} 2y - 6x - 18 = 0 \\ 2x - 8y + 28 = 0 \Longrightarrow 6x - 24y + 84 = 0 \end{array}\right\} 2y - 18 - 24y + 84 = 0.$$

$22y = 66 \Longrightarrow y = 3 \Longrightarrow 2x - 24 + 28 = 0 \Longrightarrow x = -2.$

Top is $\boxed{\text{3 miles north, 2 miles west, of South Hadley.}}$

(b) Putting in $x = -2$, $y = 3$:

$h = 10(-12 - 12 - 36 + 36 + 84 + 12) = \boxed{720\text{ ft.}}$

(c) Putting in $x = 1$, $y = 1$: $\nabla h = 10[(2 - 6 - 18)\,\hat{\mathbf{x}} + (2 - 8 + 28)\,\hat{\mathbf{y}}] = 10(-22\,\hat{\mathbf{x}} + 22\,\hat{\mathbf{y}}) = 220(-\,\hat{\mathbf{x}} + \hat{\mathbf{y}})$.

$|\nabla h| = 220\sqrt{2} \approx \boxed{311\text{ ft/mile;}}$ direction: $\boxed{\text{northwest.}}$

Problem 1.13

$$\boldsymbol{\mathscr{r}} = (x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}; \quad \mathscr{r} = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}.$$

(a) $\boldsymbol{\nabla}(\mathscr{r}^{\,2}) = \frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()\,\hat{\mathbf{z}} = 2(x-x')\,\hat{\mathbf{x}} + 2(y-y')\,\hat{\mathbf{y}} + 2(z-z')\,\hat{\mathbf{z}} = 2\,\boldsymbol{\mathscr{r}}$.

(b) $\boldsymbol{\nabla}(\frac{1}{\mathscr{r}}) = \frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]^{-\frac{1}{2}}\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()^{-\frac{1}{2}}\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()^{-\frac{1}{2}}\,\hat{\mathbf{z}}$
$= -\frac{1}{2}()^{-\frac{3}{2}}2(x-x')\,\hat{\mathbf{x}} - \frac{1}{2}()^{-\frac{3}{2}}2(y-y')\,\hat{\mathbf{y}} - \frac{1}{2}()^{-\frac{3}{2}}2(z-z')\,\hat{\mathbf{z}}$
$= -()^{-\frac{3}{2}}[(x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}] = -(1/\mathscr{r}^{\,3})\boldsymbol{\mathscr{r}} = -(1/\mathscr{r}^{\,2})\hat{\boldsymbol{\mathscr{r}}}$.

(c) $\frac{\partial}{\partial x}(\mathscr{r}^{\,n}) = n\,\mathscr{r}^{\,n-1}\frac{\partial \mathscr{r}}{\partial x} = n\,\mathscr{r}^{\,n-1}(\frac{1}{2}\frac{1}{\mathscr{r}}2\,\mathscr{r}_x) = n\,\mathscr{r}^{\,n-1}\hat{\boldsymbol{\mathscr{r}}}_x$, so $\boxed{\boldsymbol{\nabla}(\mathscr{r}^{\,n}) = n\,\mathscr{r}^{\,n-1}\,\hat{\boldsymbol{\mathscr{r}}}}$

Problem 1.14

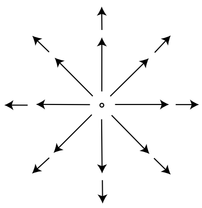
$\overline{y} = +y\,\cos\phi + z\,\sin\phi$; multiply by $\sin\phi$: $\overline{y}\sin\phi = +y\,\sin\phi\cos\phi + z\,\sin^2\phi$.
$\overline{z} = -y\,\sin\phi + z\,\cos\phi$; multiply by $\cos\phi$: $\overline{z}\cos\phi = -y\,\sin\phi\cos\phi + z\,\cos^2\phi$.
Add: $\overline{y}\sin\phi + \overline{z}\cos\phi = z(\sin^2\phi + \cos^2\phi) = z$. Likewise, $\overline{y}\cos\phi - \overline{z}\sin\phi = y$.
So $\frac{\partial y}{\partial \overline{y}} = \cos\phi$; $\frac{\partial y}{\partial \overline{z}} = -\sin\phi$; $\frac{\partial z}{\partial \overline{y}} = \sin\phi$; $\frac{\partial z}{\partial \overline{z}} = \cos\phi$. Therefore

$$\left.\begin{array}{l} \overline{(\boldsymbol{\nabla} f)}_y = \frac{\partial f}{\partial \overline{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \overline{y}} = +\cos\phi(\boldsymbol{\nabla} f)_y + \sin\phi(\boldsymbol{\nabla} f)_z \\ \overline{(\boldsymbol{\nabla} f)}_z = \frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \overline{z}} = -\sin\phi(\boldsymbol{\nabla} f)_y + \cos\phi(\boldsymbol{\nabla} f)_z \end{array}\right\} \text{So } \boldsymbol{\nabla} f \text{ transforms as a vector.} \quad \text{qed}$$

Problem 1.15

$$\boxed{\begin{array}{l} (a)\boldsymbol{\nabla\cdot}\mathbf{v}_a = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) = 2x + 0 - 2x = 0. \\ (b)\boldsymbol{\nabla\cdot}\mathbf{v}_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3xz) = y + 2z + 3x. \\ (c)\boldsymbol{\nabla\cdot}\mathbf{v}_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy+z^2) + \frac{\partial}{\partial z}(2yz) = 0 + (2x) + (2y) = 2(x+y) \end{array}}$$

Problem 1.16



$$\boldsymbol{\nabla\cdot}\mathbf{v} = \frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3}) = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$
$$+ \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$
$$= ()^{-\frac{3}{2}} + x(-3/2)()^{-\frac{5}{2}}2x + ()^{-\frac{3}{2}} + y(-3/2)()^{-\frac{5}{2}}2y + ()^{-\frac{3}{2}}$$
$$+ z(-3/2)()^{-\frac{5}{2}}2z = 3r^{-3} - 3r^{-5}(x^2+y^2+z^2) = 3r^{-3} - 3r^{-3} = 0.$$

This conclusion is surprising, because, from the diagram, this vector field is obviously diverging away from the origin. How, then, can $\boldsymbol{\nabla\cdot}\mathbf{v} = 0$? The answer is that $\boldsymbol{\nabla\cdot}\mathbf{v} = 0$ everywhere *except* at the origin, but at the origin our calculation is no good, since $r = 0$, and the expression for $\mathbf{v}$ blows up. In fact, $\boldsymbol{\nabla\cdot}\mathbf{v}$ is *infinite* at that one point, and zero elsewhere, as we shall see in Sect. 1.5.

Problem 1.17

$\overline{v}_y = \cos\phi\,v_y + \sin\phi\,v_z$; $\overline{v}_z = -\sin\phi\,v_y + \cos\phi\,v_z$.

$\frac{\partial \overline{v}_y}{\partial \overline{y}} = \frac{\partial v_y}{\partial \overline{y}}\cos\phi + \frac{\partial v_z}{\partial \overline{y}}\sin\phi = \left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \overline{y}}\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \overline{y}}\right)\sin\phi$. Use result in Prob. 1.14:
$= \left(\frac{\partial v_y}{\partial y}\cos\phi + \frac{\partial v_y}{\partial z}\sin\phi\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi\right)\sin\phi$.

$\frac{\partial \overline{v}_z}{\partial \overline{z}} = -\frac{\partial v_y}{\partial \overline{z}}\sin\phi + \frac{\partial v_z}{\partial \overline{z}}\cos\phi = -\left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \overline{z}}\right)\sin\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \overline{z}}\right)\cos\phi$
$= -\left(-\frac{\partial v_y}{\partial y}\sin\phi + \frac{\partial v_y}{\partial z}\cos\phi\right)\sin\phi + \left(-\frac{\partial v_z}{\partial y}\sin\phi + \frac{\partial v_z}{\partial z}\cos\phi\right)\cos\phi$. So

$$\frac{\partial \bar{v}_y}{\partial \bar{y}} + \frac{\partial \bar{v}_z}{\partial \bar{z}} = \frac{\partial v_y}{\partial y}\cos^2\phi + \frac{\partial v_y}{\partial z}\sin\phi\cos\phi + \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\sin^2\phi + \frac{\partial v_y}{\partial y}\sin^2\phi - \frac{\partial v_y}{\partial z}\sin\phi\cos\phi$$
$$-\frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\cos^2\phi$$
$$= \frac{\partial v_y}{\partial y}\left(\cos^2\phi + \sin^2\phi\right) + \frac{\partial v_z}{\partial z}\left(\sin^2\phi + \cos^2\phi\right) = \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \checkmark$$

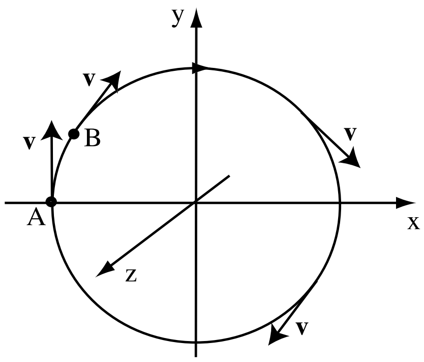
Problem 1.18

(a) $$\boldsymbol{\nabla}\times\mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = \hat{\mathbf{x}}(0-6xz) + \hat{\mathbf{y}}(0+2z) + \hat{\mathbf{z}}(3z^2-0) = \boxed{-6xz\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{y}} + 3z^2\,\hat{\mathbf{z}}.}$$

(b) $$\boldsymbol{\nabla}\times\mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = \hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x) = \boxed{-2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}.}$$

(c) $$\boldsymbol{\nabla}\times\mathbf{v}_c = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy+z^2) & 2yz \end{vmatrix} = \hat{\mathbf{x}}(2z-2z) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(2y-2y) = \boxed{\mathbf{0}.}$$

Problem 1.19



As we go from point A to point B (9 o'clock to 10 o'clock), x increases, y increases, v_x increases, and v_y decreases, so $\partial v_x/\partial y > 0$, while $\partial v_y/\partial y < 0$. On the circle, $v_z = 0$, and there is no dependence on z, so Eq. 1.41 says

$$\boldsymbol{\nabla}\times\mathbf{v} = \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

points in the $\boxed{\text{negative } z \text{ direction}}$ (into the page), as the right hand rule would suggest. (Pick any other nearby points on the circle and you will come to the same conclusion.) [I'm sorry, but I cannot remember who suggested this cute illustration.]

Problem 1.20

$\mathbf{v} = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$; or $\mathbf{v} = yz\,\hat{\mathbf{x}} + xz\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$; or $\mathbf{v} = (3x^2z - z^3)\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}} + (x^3 - 3xz^2)\,\hat{\mathbf{z}}$;
or $\mathbf{v} = (\sin x)(\cosh y)\,\hat{\mathbf{x}} - (\cos x)(\sinh y)\,\hat{\mathbf{y}}$; etc.

Problem 1.21

(i) $$\boldsymbol{\nabla}(fg) = \frac{\partial(fg)}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial(fg)}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial(fg)}{\partial z}\,\hat{\mathbf{z}} = \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\hat{\mathbf{x}} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\hat{\mathbf{y}} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\hat{\mathbf{z}}$$
$$= f\left(\frac{\partial g}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial g}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial g}{\partial z}\,\hat{\mathbf{z}}\right) + g\left(\frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}\right) = f(\boldsymbol{\nabla}g) + g(\boldsymbol{\nabla}f). \quad \text{qed}$$

(iv) $$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x)$$
$$= A_y\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_y}{\partial x} - A_z\frac{\partial B_y}{\partial x} - B_y\frac{\partial A_z}{\partial x} + A_z\frac{\partial B_x}{\partial y} + B_x\frac{\partial A_z}{\partial y} - A_x\frac{\partial B_z}{\partial y} - B_z\frac{\partial A_x}{\partial y}$$
$$+A_x\frac{\partial B_y}{\partial z} + B_y\frac{\partial A_x}{\partial z} - A_y\frac{\partial B_x}{\partial z} - B_x\frac{\partial A_y}{\partial z}$$
$$= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)$$
$$-A_y\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}). \quad \text{qed}$$

(v) $$\boldsymbol{\nabla}\times(f\mathbf{A}) = \left(\frac{\partial(fA_z)}{\partial y} - \frac{\partial(fA_y)}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial(fA_x)}{\partial z} - \frac{\partial(fA_z)}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial(fA_y)}{\partial x} - \frac{\partial(fA_x)}{\partial y}\right)\hat{\mathbf{z}}$$

$$
\begin{aligned}
&= \left(f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right)\hat{\mathbf{x}} + \left(f\frac{\partial A_x}{\partial z} + A_x\frac{\partial f}{\partial z} - f\frac{\partial A_z}{\partial x} - A_z\frac{\partial f}{\partial x}\right)\hat{\mathbf{y}} \\
&\qquad + \left(f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right)\hat{\mathbf{z}} \\
&= f\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}\right] \\
&\qquad - \left[\left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial y}\right)\hat{\mathbf{x}} + \left(A_z\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial z}\right)\hat{\mathbf{y}} + \left(A_x\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial x}\right)\hat{\mathbf{z}}\right] \\
&= f\,(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla} f). \quad \text{qed}
\end{aligned}
$$

Problem 1.22

(a)
$$
\begin{aligned}
(\mathbf{A}\cdot\nabla)\,\mathbf{B} &= \left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right)\hat{\mathbf{x}} + \left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)\hat{\mathbf{y}} \\
&\quad + \left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}.
\end{aligned}
$$

(b) $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}$. Let's just do the x component.

$$
\begin{aligned}
[(\hat{\mathbf{r}}\cdot\nabla)\hat{\mathbf{r}}]_x &= \frac{1}{\sqrt{}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{x}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{1}{r}\left\{x\left[\frac{1}{\sqrt{}} + x(-\tfrac{1}{2})\frac{1}{(\sqrt{})^3}2x\right] + yx\left[-\tfrac{1}{2}\frac{1}{(\sqrt{})^3}2y\right] + zx\left[-\tfrac{1}{2}\frac{1}{(\sqrt{})^3}2z\right]\right\} \\
&= \frac{1}{r}\left\{\frac{x}{r} - \frac{1}{r^3}\left(x^3 + xy^2 + xz^2\right)\right\} = \frac{1}{r}\left\{\frac{x}{r} - \frac{x}{r^3}\left(x^2+y^2+z^2\right)\right\} = \frac{1}{r}\left(\frac{x}{r} - \frac{x}{r}\right) = 0.
\end{aligned}
$$

Same goes for the other components. Hence: $\boxed{(\hat{\mathbf{r}}\cdot\nabla)\,\hat{\mathbf{r}} = \mathbf{0}}$.

(c)
$$
\begin{aligned}
(\mathbf{v}_a\cdot\nabla)\,\mathbf{v}_b &= \left(x^2\frac{\partial}{\partial x} + 3xz^2\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}\right)(xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3xz\,\hat{\mathbf{z}}) \\
&= x^2\,(y\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}) + 3xz^2\,(x\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) - 2xz\,(0\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3x\,\hat{\mathbf{z}}) \\
&= \left(x^2y + 3x^2z^2\right)\hat{\mathbf{x}} + \left(6xz^3 - 4xyz\right)\hat{\mathbf{y}} + \left(3x^2z - 6x^2z\right)\hat{\mathbf{z}} \\
&= \boxed{x^2\left(y+3z^2\right)\hat{\mathbf{x}} + 2xz\left(3z^2-2y\right)\hat{\mathbf{y}} - 3x^2z\,\hat{\mathbf{z}}}
\end{aligned}
$$

Problem 1.23

(ii) $[\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x = \frac{\partial}{\partial x}(A_xB_x + A_yB_y + A_zB_z) = \frac{\partial A_x}{\partial x}B_x + A_x\frac{\partial B_x}{\partial x} + \frac{\partial A_y}{\partial x}B_y + A_y\frac{\partial B_y}{\partial x} + \frac{\partial A_z}{\partial x}B_z + A_z\frac{\partial B_z}{\partial x}$

$$
\begin{aligned}
[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})]_x &= A_y(\boldsymbol{\nabla}\times\mathbf{B})_z - A_z(\boldsymbol{\nabla}\times\mathbf{B})_y = A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) \\
[\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})]_x &= B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - B_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\
[(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}]_x &= \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_x = A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z} \\
[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}]_x &= B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}
\end{aligned}
$$

So $[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}]_x$

$$
\begin{aligned}
&= A_y\frac{\partial B_y}{\partial x} - A_y\frac{\partial B_x}{\partial y} - A_z\frac{\partial B_x}{\partial z} + A_z\frac{\partial B_z}{\partial x} + B_y\frac{\partial A_y}{\partial x} - B_y\frac{\partial A_x}{\partial y} - B_z\frac{\partial A_x}{\partial z} + B_z\frac{\partial A_z}{\partial x} \\
&\quad + A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z} + B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z} \\
&= B_x\frac{\partial A_x}{\partial x} + A_x\frac{\partial B_x}{\partial x} + B_y\left(\frac{\partial A_y}{\partial x} - \cancel{\frac{\partial A_x}{\partial y}} + \cancel{\frac{\partial A_x}{\partial y}}\right) + A_y\left(\frac{\partial B_y}{\partial x} - \cancel{\frac{\partial B_x}{\partial y}} + \cancel{\frac{\partial B_x}{\partial y}}\right) \\
&\quad + B_z\left(-\cancel{\frac{\partial A_x}{\partial z}} + \frac{\partial A_z}{\partial x} + \cancel{\frac{\partial A_x}{\partial z}}\right) + A_z\left(-\cancel{\frac{\partial B_x}{\partial z}} + \frac{\partial B_z}{\partial x} + \cancel{\frac{\partial B_x}{\partial z}}\right) \\
&= [\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x \text{ (same for } y \text{ and } z\text{)}
\end{aligned}
$$

(vi)
$$
\begin{aligned}
[\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x &= \frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})_z - \frac{\partial}{\partial z}(\mathbf{A}\times\mathbf{B})_y = \frac{\partial}{\partial y}(A_xB_y - A_yB_x) - \frac{\partial}{\partial z}(A_zB_x - A_xB_z) \\
&= \frac{\partial A_x}{\partial y}B_y + A_x\frac{\partial B_y}{\partial y} - \frac{\partial A_y}{\partial y}B_x - A_y\frac{\partial B_x}{\partial y} - \frac{\partial A_z}{\partial z}B_x - A_z\frac{\partial B_x}{\partial z} + \frac{\partial A_x}{\partial z}B_z + A_x\frac{\partial B_z}{\partial z}
\end{aligned}
$$

$[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})]_x$

$$
= B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z} - A_x\frac{\partial B_x}{\partial x} - A_y\frac{\partial B_x}{\partial y} - A_z\frac{\partial B_x}{\partial z} + A_x\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) - B_x\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)
$$

$$= B_y\frac{\partial A_x}{\partial y} + A_x\left(-\cancel{\frac{\partial B_x}{\partial x}} + \cancel{\frac{\partial B_x}{\partial x}} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) + B_x\left(\cancel{\frac{\partial A_x}{\partial x}} - \cancel{\frac{\partial A_x}{\partial x}} - \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial z}\right)$$
$$+ A_y\left(-\frac{\partial B_x}{\partial y}\right) + A_z\left(-\frac{\partial B_x}{\partial z}\right) + B_z\left(\frac{\partial A_x}{\partial z}\right)$$
$$= [\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x \text{ (same for } y \text{ and } z)$$

Problem 1.24

$$\begin{aligned}\boldsymbol{\nabla}(f/g) &= \frac{\partial}{\partial x}(f/g)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(f/g)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(f/g)\,\hat{\mathbf{z}} \\ &= \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\,\hat{\mathbf{x}} + \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\,\hat{\mathbf{y}} + \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\,\hat{\mathbf{z}} \\ &= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}\right) - f\left(\frac{\partial g}{\partial x}\hat{\mathbf{x}} + \frac{\partial g}{\partial y}\hat{\mathbf{y}} + \frac{\partial g}{\partial z}\hat{\mathbf{z}}\right)\right] = \frac{g\nabla f - f\nabla g}{g^2}. \quad \text{qed}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{A}/g) &= \frac{\partial}{\partial x}(A_x/g) + \frac{\partial}{\partial y}(A_y/g) + \frac{\partial}{\partial z}(A_z/g) \\ &= \frac{g\frac{\partial A_x}{\partial x} - A_x\frac{\partial g}{\partial x}}{g^2} + \frac{g\frac{\partial A_y}{\partial y} - A_y\frac{\partial g}{\partial y}}{g^2} + \frac{g\frac{\partial A_z}{\partial z} - A_z\frac{\partial g}{\partial x}}{g^2} \\ &= \frac{1}{g^2}\left[g\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} + A_z\frac{\partial g}{\partial z}\right)\right] = \frac{g\nabla\cdot\mathbf{A} - \mathbf{A}\cdot\nabla g}{g^2}. \quad \text{qed}\end{aligned}$$

$$\begin{aligned}[\boldsymbol{\nabla}\times(\mathbf{A}/g)]_x &= \frac{\partial}{\partial y}(A_z/g) - \frac{\partial}{\partial z}(A_y/g) \\ &= \frac{g\frac{\partial A_z}{\partial y} - A_z\frac{\partial g}{\partial y}}{g^2} - \frac{g\frac{\partial A_y}{\partial z} - A_y\frac{\partial g}{\partial z}}{g^2} \\ &= \frac{1}{g^2}\left[g\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \left(A_z\frac{\partial g}{\partial y} - A_y\frac{\partial g}{\partial z}\right)\right] \\ &= \frac{g(\nabla\times\mathbf{A})_x + (\mathbf{A}\times\nabla g)_x}{g^2} \text{ (same for } y \text{ and } z). \quad \text{qed}\end{aligned}$$

Problem 1.25

(a) $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0\end{vmatrix} = \hat{\mathbf{x}}(6xz) + \hat{\mathbf{y}}(9zy) + \hat{\mathbf{z}}(-2x^2 - 6y^2)$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9zy) + \frac{\partial}{\partial z}(-2x^2 - 6y^2) = 6z + 9z + 0 = 15z$$
$$\boldsymbol{\nabla}\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(3z)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(x)\right) = 0;\ \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$$
$$\boldsymbol{\nabla}\times\mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2x)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(3y) - \frac{\partial}{\partial x}(0)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(3y)\right) = -5\,\hat{\mathbf{z}};\ \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = -15z$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) \stackrel{?}{=} \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = 0 - (-15z) = 15z.\ \checkmark$$

(b) $\mathbf{A}\cdot\mathbf{B} = 3xy - 4xy = -xy$; $\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \boldsymbol{\nabla}(-xy) = \hat{\mathbf{x}}\frac{\partial}{\partial x}(-xy) + \hat{\mathbf{y}}\frac{\partial}{\partial y}(-xy) = -y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}}$

$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5\end{vmatrix} = \hat{\mathbf{x}}(-10y) + \hat{\mathbf{y}}(5x);\ \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \mathbf{0}$$

$$(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} = \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right)(3y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}}) = \hat{\mathbf{x}}(6y) + \hat{\mathbf{y}}(-2x)$$
$$(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} = \left(3y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}\right)(x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}) = \hat{\mathbf{x}}(3y) + \hat{\mathbf{y}}(-4x)$$

$$\begin{aligned}&\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} \\ &\quad = -10y\,\hat{\mathbf{x}} + 5x\,\hat{\mathbf{y}} + 6y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}} + 3y\,\hat{\mathbf{x}} - 4x\,\hat{\mathbf{y}} = -y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}} = \boldsymbol{\nabla}\cdot(\mathbf{A}\cdot\mathbf{B}).\ \checkmark\end{aligned}$$

(c) $$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(-2x^2 - 6y^2) - \frac{\partial}{\partial z}(9zy)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(6xz) - \frac{\partial}{\partial x}(-2x^2 - 6y^2)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(9zy) - \frac{\partial}{\partial y}(6xz)\right) \\ &= \hat{\mathbf{x}}(-12y - 9y) + \hat{\mathbf{y}}(6x + 4x) + \hat{\mathbf{z}}(0) = -21y\,\hat{\mathbf{x}} + 10x\,\hat{\mathbf{y}}\end{aligned}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) = 1 + 2 + 3 = 6;\ \boldsymbol{\nabla}\cdot\mathbf{B} = \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(-2x) = 0$$

$(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) = 3y\,\hat{\mathbf{x}} - 4x\,\hat{\mathbf{y}} - 6y\,\hat{\mathbf{x}} + 2x\,\hat{\mathbf{y}} - 18y\,\hat{\mathbf{x}} + 12x\,\hat{\mathbf{y}} = -21y\,\hat{\mathbf{x}} + 10x\,\hat{\mathbf{y}}$
$= \boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})$. ✓

Problem 1.26

(a) $\frac{\partial^2 T_a}{\partial x^2} = 2;\ \frac{\partial^2 T_a}{\partial y^2} = \frac{\partial^2 T_a}{\partial z^2} = 0 \ \Rightarrow \boxed{\nabla^2 T_a = 2.}$

(b) $\frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -T_b \ \Rightarrow \boxed{\nabla^2 T_b = -3T_b = -3\sin x \sin y \sin z.}$

(c) $\frac{\partial^2 T_c}{\partial x^2} = 25T_c\ ;\ \frac{\partial^2 T_c}{\partial y^2} = -16T_c\ ;\ \frac{\partial^2 T_c}{\partial z^2} = -9T_c \ \Rightarrow \boxed{\nabla^2 T_c = 0.}$

(d)
$$\left.\begin{array}{l} \frac{\partial^2 v_x}{\partial x^2} = 2\ ;\ \frac{\partial^2 v_x}{\partial y^2} = \frac{\partial^2 v_x}{\partial z^2} = 0 \ \Rightarrow\ \nabla^2 v_x = 2 \\ \frac{\partial^2 v_y}{\partial x^2} = \frac{\partial^2 v_y}{\partial y^2} = 0\ ;\ \frac{\partial^2 v_y}{\partial z^2} = 6x \ \Rightarrow\ \nabla^2 v_y = 6x \\ \frac{\partial^2 v_z}{\partial x^2} = \frac{\partial^2 v_z}{\partial y^2} = \frac{\partial^2 v_z}{\partial z^2} = 0 \ \Rightarrow\ \nabla^2 v_z = 0 \end{array}\right\} \boxed{\nabla^2 \mathbf{v} = 2\,\hat{\mathbf{x}} + 6x\,\hat{\mathbf{y}}.}$$

Problem 1.27

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$
$$= \left(\frac{\partial^2 v_z}{\partial x\,\partial y} - \frac{\partial^2 v_z}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 v_x}{\partial y\,\partial z} - \frac{\partial^2 v_x}{\partial z\,\partial y}\right) + \left(\frac{\partial^2 v_y}{\partial z\,\partial x} - \frac{\partial^2 v_y}{\partial x\,\partial z}\right) = 0, \text{ by equality of cross-derivatives.}$$

From Prob. 1.18: $\boldsymbol{\nabla}\times\mathbf{v}_a = -6xz\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{y}} + 3z^2\,\hat{\mathbf{z}} \ \Rightarrow\ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}_a) = \frac{\partial}{\partial x}(-6xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3z^2) = -6z + 6z = 0.$

Problem 1.28

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}t) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial^2 t}{\partial y\,\partial z} - \frac{\partial^2 t}{\partial z\,\partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 t}{\partial z\,\partial x} - \frac{\partial^2 t}{\partial x\,\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 t}{\partial x\,\partial y} - \frac{\partial^2 t}{\partial y\,\partial x}\right)$$
$= 0$, by equality of cross-derivatives.

In Prob. 1.11(b), $\boldsymbol{\nabla} f = 2xy^3z^4\,\hat{\mathbf{x}} + 3x^2y^2z^4\,\hat{\mathbf{y}} + 4x^2y^3z^3\,\hat{\mathbf{z}}$, so

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$
$$= \hat{\mathbf{x}}(3\cdot 4x^2y^2z^3 - 4\cdot 3x^2y^2z^3) + \hat{\mathbf{y}}(4\cdot 2xy^3z^3 - 2\cdot 4xy^3z^3) + \hat{\mathbf{z}}(2\cdot 3xy^2z^4 - 3\cdot 2xy^2z^4) = 0. \ ✓$$

Problem 1.29

(a) $(0,0,0) \longrightarrow (1,0,0)$. $x: 0\to 1, y = z = 0; d\mathbf{l} = dx\,\hat{\mathbf{x}}; \mathbf{v}\cdot d\mathbf{l} = x^2\,dx; \int \mathbf{v}\cdot d\mathbf{l} = \int_0^1 x^2\,dx = (x^3/3)|_0^1 = 1/3.$
$(1,0,0) \longrightarrow (1,1,0)$. $x = 1, y: 0\to 1, z = 0; d\mathbf{l} = dy\,\hat{\mathbf{y}}; \mathbf{v}\cdot d\mathbf{l} = 2yz\,dy = 0; \int \mathbf{v}\cdot d\mathbf{l} = 0.$
$(1,1,0) \longrightarrow (1,1,1)$. $x = y = 1, z: 0\to 1; d\mathbf{l} = dz\,\hat{\mathbf{z}}; \mathbf{v}\cdot d\mathbf{l} = y^2\,dz = dz; \int \mathbf{v}\cdot d\mathbf{l} = \int_0^1 dz = z|_0^1 = 1.$
Total: $\int \mathbf{v}\cdot d\mathbf{l} = (1/3) + 0 + 1 = \boxed{4/3.}$

(b) $(0,0,0) \longrightarrow (0,0,1)$. $x = y = 0, z: 0\to 1; d\mathbf{l} = dz\,\hat{\mathbf{z}}; \mathbf{v}\cdot d\mathbf{l} = y^2\,dz = 0; \int \mathbf{v}\cdot d\mathbf{l} = 0.$
$(0,0,1) \longrightarrow (0,1,1)$. $x = 0, y: 0\to 1, z = 1; d\mathbf{l} = dy\,\hat{\mathbf{y}}; \mathbf{v}\cdot d\mathbf{l} = 2yz\,dy = 2y\,dy; \int \mathbf{v}\cdot d\mathbf{l} = \int_0^1 2y\,dy = y^2|_0^1 = 1.$
$(0,1,1) \longrightarrow (1,1,1)$. $x: 0\to 1, y = z = 1; d\mathbf{l} = dx\,\hat{\mathbf{x}}; \mathbf{v}\cdot d\mathbf{l} = x^2\,dx; \int \mathbf{v}\cdot d\mathbf{l} = \int_0^1 x^2\,dx = (x^3/3)|_0^1 = 1/3.$
Total: $\int \mathbf{v}\cdot d\mathbf{l} = 0 + 1 + (1/3) = \boxed{4/3.}$

(c) $x = y = z: 0\to 1; dx = dy = dz; \mathbf{v}\cdot d\mathbf{l} = x^2\,dx + 2yz\,dy + y^2\,dz = x^2\,dx + 2x^2\,dx + x^2\,dx = 4x^2\,dx;$
$\int \mathbf{v}\cdot d\mathbf{l} = \int_0^1 4x^2\,dx = (4x^3/3)|_0^1 = \boxed{4/3.}$

(d) $\oint \mathbf{v}\cdot d\mathbf{l} = (4/3) - (4/3) = \boxed{0.}$

Problem 1.30

$x, y : 0 \to 1, z = 0; d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}; \mathbf{v}\cdot d\mathbf{a} = y(z^2-3)\,dx\,dy = -3y\,dx\,dy; \int \mathbf{v}\cdot d\mathbf{a} = -3\int_0^2 dx \int_0^2 y\,dy = -3(x|_0^2)(\frac{y^2}{2}|_0^2) = -3(2)(2) = \boxed{-12.}$ In Ex. 1.7 we got 20, for the same boundary line (the square in the xy-plane), so the answer is $\boxed{\text{no:}}$ the surface integral does *not* depend only on the boundary line. The *total* flux for the cube is $20 + 12 = \boxed{32.}$

Problem 1.31

$\int T\,d\tau = \int z^2\,dx\,dy\,dz$. You can do the integrals in any order—here it is simplest to save z for last:

$$\int z^2 \left[\int\left(\int dx\right) dy\right] dz.$$

The sloping surface is $x+y+z=1$, so the x integral is $\int_0^{(1-y-z)} dx = 1-y-z$. For a given z, y ranges from 0 to $1-z$, so the y integral is $\int_0^{(1-z)}(1-y-z)\,dy = [(1-z)y-(y^2/2)]|_0^{(1-z)} = (1-z)^2 - [(1-z)^2/2] = (1-z)^2/2 = (1/2) - z + (z^2/2)$. Finally, the z integral is $\int_0^1 z^2(\frac{1}{2} - z + \frac{z^2}{2})\,dz = \int_0^1(\frac{z^2}{2} - z^3 + \frac{z^4}{2})\,dz = (\frac{z^3}{6} - \frac{z^4}{4} + \frac{z^5}{10})|_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \boxed{1/60.}$

Problem 1.32

$T(\mathbf{b}) = 1+4+2 = 7;\ T(\mathbf{a}) = 0.\ \Rightarrow \boxed{T(\mathbf{b}) - T(\mathbf{a}) = 7.}$

$\boldsymbol{\nabla} T = (2x+4y)\hat{\mathbf{x}} + (4x+2z^3)\hat{\mathbf{y}} + (6yz^2)\hat{\mathbf{z}};\ \boldsymbol{\nabla} T\cdot d\mathbf{l} = (2x+4y)dx + (4x+2z^3)dy + (6yz^2)dz$

$$\left.\begin{array}{l}\text{(a) Segment 1: } x: 0\to 1,\ y=z=dy=dz=0.\ \int \boldsymbol{\nabla} T\cdot d\mathbf{l} = \int_0^1 (2x)\,dx = x^2\big|_0^1 = 1.\\ \quad\ \text{Segment 2: } y: 0\to 1,\ x=1,\ z=0,\ dx=dz=0.\ \int \boldsymbol{\nabla} T\cdot d\mathbf{l} = \int_0^1 (4)\,dy = 4y\big|_0^1 = 4.\\ \quad\ \text{Segment 3: } z: 0\to 1,\ x=y=1,\ dx=dy=0.\ \int \boldsymbol{\nabla} T\cdot d\mathbf{l} = \int_0^1 (6z^2)\,dz = 2z^3\big|_0^1 = 2.\end{array}\right\} \int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} T\cdot d\mathbf{l} = 7.\ \checkmark$$

$$\left.\begin{array}{l}\text{(b) Segment 1: } z: 0\to 1,\ x=y=dx=dy=0.\ \int \boldsymbol{\nabla} T\cdot d\mathbf{l} = \int_0^1 (0)\,dz = 0.\\ \quad\ \text{Segment 2: } y: 0\to 1,\ x=0,\ z=1,\ dx=dz=0.\ \int \boldsymbol{\nabla} T\cdot d\mathbf{l} = \int_0^1 (2)\,dy = 2y\big|_0^1 = 2.\\ \quad\ \text{Segment 3: } x: 0\to 1,\ y=z=1,\ dy=dz=0.\ \int \boldsymbol{\nabla} T\cdot d\mathbf{l} = \int_0^1 (2x+4)\,dx\\ \qquad\qquad = (x^2+4x)\big|_0^1 = 1+4 = 5.\end{array}\right\} \int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} T\cdot d\mathbf{l} = 7.\ \checkmark$$

(c) $x: 0\to 1,\ y = x,\ z = x^2,\ dy = dx,\ dz = 2x\,dx.$

$\boldsymbol{\nabla} T\cdot d\mathbf{l} = (2x+4x)dx + (4x+2x^6)dx + (6xx^4)2x\,dx = (10x+14x^6)dx.$

$\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} T\cdot d\mathbf{l} = \int_0^1 (10x+14x^6)dx = (5x^2+2x^7)\big|_0^1 = 5+2 = 7.\ \checkmark$

Problem 1.33

$\boldsymbol{\nabla}\cdot\mathbf{v} = y + 2z + 3x$

$$\begin{aligned}\int(\boldsymbol{\nabla}\cdot\mathbf{v})d\tau &= \int(y+2z+3x)\,dx\,dy\,dz = \iint\left\{\int_0^2 (y+2z+3x)\,dx\right\}dy\,dz\\ &\qquad\hookrightarrow \left[(y+2z)x + \tfrac{3}{2}x^2\right]_0^2 = 2(y+2z)+6\\ &= \int\left\{\int_0^2 (2y+4z+6)dy\right\}dz\\ &\qquad\hookrightarrow \left[y^2 + (4z+6)y\right]_0^2 = 4 + 2(4z+6) = 8z+16\\ &= \int_0^2 (8z+16)dz = (4z^2+16z)\big|_0^2 = 16+32 = \boxed{48.}\end{aligned}$$

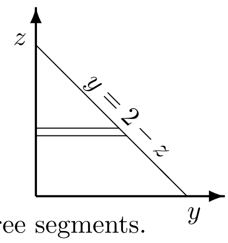
Numbering the surfaces as in Fig. 1.29:

(i) $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}, x = 2.$ $\mathbf{v}\cdot d\mathbf{a} = 2y\,dy\,dz.$ $\int \mathbf{v}\cdot d\mathbf{a} = \iint 2y\,dy\,dz = 2y^2\Big|_0^2 = 8.$
(ii) $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}, x = 0.$ $\mathbf{v}\cdot d\mathbf{a} = 0.$ $\int \mathbf{v}\cdot d\mathbf{a} = 0.$
(iii) $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}}, y = 2.$ $\mathbf{v}\cdot d\mathbf{a} = 4z\,dx\,dz.$ $\int \mathbf{v}\cdot d\mathbf{a} = \iint 4z\,dx\,dz = 16.$
(iv) $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}}, y = 0.$ $\mathbf{v}\cdot d\mathbf{a} = 0.$ $\int \mathbf{v}\cdot d\mathbf{a} = 0.$
(v) $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}, z = 2.$ $\mathbf{v}\cdot d\mathbf{a} = 6x\,dx\,dy.$ $\int \mathbf{v}\cdot d\mathbf{a} = 24.$
(vi) $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}}, z = 0.$ $\mathbf{v}\cdot d\mathbf{a} = 0.$ $\int \mathbf{v}\cdot d\mathbf{a} = 0.$
$\Rightarrow \int \mathbf{v}\cdot d\mathbf{a} = 8 + 16 + 24 = 48$ ✓

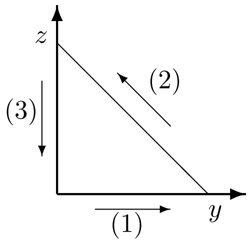
Problem 1.34

$\nabla\times\mathbf{v} = \hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x) = -2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}.$
$d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, if we agree that the path integral shall run counterclockwise. So $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = -2y\,dy\,dz.$

$$\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int\left\{\underbrace{\int_0^{2-z}(-2y)dy}_{y^2\big|_0^{2-z} = -(2-z)^2}\right\}dz$$
$$= -\int_0^2(4-4z+z^2)dz = -\left(4z - 2z^2 + \frac{z^3}{3}\right)\Big|_0^2$$
$$= -\left(8 - 8 + \tfrac{8}{3}\right) = \boxed{-\tfrac{8}{3}}$$



Meanwhile, $\mathbf{v}\cdot d\mathbf{l} = (xy)dx + (2yz)dy + (3zx)dz$. There are three segments.



(1) $x = z = 0;$ $dx = dz = 0.$ $y : 0 \to 2.$ $\int \mathbf{v}\cdot d\mathbf{l} = 0.$
(2) $x = 0;$ $z = 2 - y;$ $dx = 0,$ $dz = -dy,$ $y : 2 \to 0.$ $\mathbf{v}\cdot d\mathbf{l} = 2yz\,dy.$
$\int \mathbf{v}\cdot d\mathbf{l} = \int_2^0 2y(2-y)dy = -\int_0^2(4y - 2y^2)dy = -\left(2y^2 - \tfrac{2}{3}y^3\right)\Big|_0^2 = -\left(8 - \tfrac{2}{3}\cdot 8\right) = -\tfrac{8}{3}.$
(3) $x = y = 0;$ $dx = dy = 0;$ $z : 2 \to 0.$ $\mathbf{v}\cdot d\mathbf{l} = 0.$ $\int \mathbf{v}\cdot d\mathbf{l} = 0.$ So $\oint \mathbf{v}\cdot d\mathbf{l} = -\tfrac{8}{3}.$ ✓

Problem 1.35

By Corollary 1, $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$ should equal $\tfrac{4}{3}$. $\nabla\times\mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\,\hat{\mathbf{z}}.$

(i) $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}},$ $x = 1;$ $y, z : 0 \to 1.$ $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = (4z^2 - 2)dy\,dz;$ $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int_0^1(4z^2 - 2)dz$ $= \left(\tfrac{4}{3}z^3 - 2z\right)\Big|_0^1 = \tfrac{4}{3} - 2 = -\tfrac{2}{3}.$
(ii) $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}},$ $z = 0;$ $x, y : 0 \to 1.$ $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0;$ $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0.$
(iii) $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}},$ $y = 1;$ $x, z : 0 \to 1.$ $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0;$ $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0.$
(iv) $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}},$ $y = 0;$ $x, z : 0 \to 1.$ $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0;$ $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0.$
(v) $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}},$ $z = 1;$ $x, y : 0 \to 1.$ $(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 2\,dx\,dy;$ $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 2.$

$\Rightarrow \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = -\tfrac{2}{3} + 2 = \tfrac{4}{3}.$ ✓

Problem 1.36

(a) Use the product rule $\boldsymbol{\nabla}\times(f\mathbf{A}) = f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}f)$:

$$\int_{\mathcal{S}} f(\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{a} = \int_{\mathcal{S}} \boldsymbol{\nabla}\times(f\mathbf{A})\cdot d\mathbf{a} + \int_{\mathcal{S}} [\mathbf{A}\times(\boldsymbol{\nabla}f)]\cdot d\mathbf{a} = \oint_{\mathcal{P}} f\mathbf{A}\cdot d\mathbf{l} + \int_{\mathcal{S}} [\mathbf{A}\times(\boldsymbol{\nabla}f)]\cdot d\mathbf{a}. \quad \text{qed}$$

(I used Stokes' theorem in the last step.)

(b) Use the product rule $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$:

$$\int_{\mathcal{V}} \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})d\tau = \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B})\,d\tau + \int_{\mathcal{V}} \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,d\tau = \oint_{\mathcal{S}} (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a} + \int_{\mathcal{V}} \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,d\tau. \quad \text{qed}$$

(I used the divergence theorem in the last step.)

Problem 1.37 $\boxed{r = \sqrt{x^2+y^2+z^2}; \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right); \quad \phi = \tan^{-1}\left(\frac{y}{x}\right).}$

Problem 1.38

There are many ways to do this one—probably the most illuminating way is to work it out by trigonometry from Fig. 1.36. The most systematic approach is to study the expression:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}.$$

If I only vary r slightly, then $d\mathbf{r} = \frac{\partial}{\partial r}(\mathbf{r})dr$ is a short vector pointing in the direction of increase in r. To make it a unit vector, I must divide by its length. Thus:

$$\hat{\mathbf{r}} = \frac{\frac{\partial\mathbf{r}}{\partial r}}{\left|\frac{\partial\mathbf{r}}{\partial r}\right|}; \ \hat{\boldsymbol{\theta}} = \frac{\frac{\partial\mathbf{r}}{\partial\theta}}{\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|}; \ \hat{\boldsymbol{\phi}} = \frac{\frac{\partial\mathbf{r}}{\partial\phi}}{\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|}.$$

$\frac{\partial\mathbf{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$; $\left|\frac{\partial\mathbf{r}}{\partial r}\right|^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1.$

$\frac{\partial\mathbf{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{\mathbf{x}} + r\cos\theta\sin\phi\,\hat{\mathbf{y}} - r\sin\theta\,\hat{\mathbf{z}}$; $\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2.$

$\frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\hat{\mathbf{x}} + r\sin\theta\cos\phi\,\hat{\mathbf{y}}$; $\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta.$

$$\Rightarrow \boxed{\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}.\\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}.\\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.\end{aligned}}$$

Check: $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$, ✓

$\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = -\cos\theta\sin\phi\cos\phi + \cos\theta\sin\phi\cos\phi = 0$, ✓ etc.

$\sin\theta\,\hat{\mathbf{r}} = \sin^2\theta\cos\phi\,\hat{\mathbf{x}} + \sin^2\theta\sin\phi\,\hat{\mathbf{y}} + \sin\theta\cos\theta\,\hat{\mathbf{z}}.$

$\cos\theta\,\hat{\boldsymbol{\theta}} = \cos^2\theta\cos\phi\,\hat{\mathbf{x}} + \cos^2\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\cos\theta\,\hat{\mathbf{z}}.$

Add these:

(1) $\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} = +\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$;

(2) $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.$

Multiply (1) by $\cos\phi$, (2) by $\sin\phi$, and subtract:

$$\boxed{\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}.}$$

Multiply (1) by $\sin\phi$, (2) by $\cos\phi$, and add:

$$\boxed{\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}.}$$

$$\begin{aligned}\cos\theta\,\hat{\mathbf{r}} &= \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \cos^2\theta\,\hat{\mathbf{z}}.\\ \sin\theta\,\hat{\boldsymbol{\theta}} &= \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin^2\theta\,\hat{\mathbf{z}}.\end{aligned}$$

Subtract these:

$$\boxed{\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.}$$

Problem 1.39

(a) $\boldsymbol{\nabla}\cdot\mathbf{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2) = \frac{1}{r^2}4r^3 = 4r$

$\int(\boldsymbol{\nabla}\cdot\mathbf{v}_1)d\tau = \int(4r)(r^2\sin\theta\,dr\,d\theta\,d\phi) = (4)\int_0^R r^3 dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi = (4)\left(\frac{R^4}{4}\right)(2)(2\pi) = \boxed{4\pi R^4}$

$\int \mathbf{v_1}\cdot d\mathbf{a} = \int(r^2\hat{\mathbf{r}})\cdot(r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}) = r^4\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi = 4\pi R^4$ ✓ (*Note:* at surface of sphere $r = R$.)

(b) $\boldsymbol{\nabla}\cdot\mathbf{v}_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0 \;\Rightarrow \boxed{\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)d\tau = 0}$

$\int \mathbf{v}_2\cdot d\mathbf{a} = \int\left(\frac{1}{r^2}\hat{\mathbf{r}}\right)(r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}) = \int\sin\theta\,d\theta\,d\phi = \boxed{4\pi.}$

They *don't* agree! The point is that this divergence is zero *except at the origin*, where it blows up, so our calculation of $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)$ is *incorrect.* The right answer is 4π.

Problem 1.40

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\\ &= \frac{1}{r^2}\,3r^2\cos\theta + \frac{1}{r\sin\theta}\,r\,2\sin\theta\cos\theta + \frac{1}{r\sin\theta}\,r\sin\theta(-\sin\phi)\\ &= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi\end{aligned}$$

$$\int(\boldsymbol{\nabla}\cdot\mathbf{v})d\tau = \int(5\cos\theta - \sin\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi = \int_0^R r^2\,dr\int_0^{\frac{\theta}{2}}\Bigg[\underbrace{\int_0^{2\pi}(5\cos\theta - \sin\phi)\,d\phi}_{\hookrightarrow 2\pi(5\cos\theta)}\Bigg]\,d\theta\sin\theta$$

$$= \left(\frac{R^3}{3}\right)(10\pi)\underbrace{\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta}_{\hookrightarrow \frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}}$$

$$= \boxed{\frac{5\pi}{3}R^3.}$$

Two surfaces—one the hemisphere: $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}};\; r = R;\; \phi: 0\to 2\pi,\; \theta: 0\to\frac{\pi}{2}$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta\,d\theta\,d\phi = R^3\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_0^{2\pi}d\phi = R^3\left(\frac{1}{2}\right)(2\pi) = \pi R^3$.

other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta\,d\phi)(+\hat{\boldsymbol{\theta}}) = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$ (here $\theta = \frac{\pi}{2}$). $r: 0\to R,\; \phi: 0\to 2\pi$.

$\int\mathbf{v}\cdot d\mathbf{a} = \int(r\sin\theta)(r\,dr\,d\phi) = \int_0^R r^2\,dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}$.

Total: $\int\mathbf{v}\cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3$. ✓

Problem 1.41 $\boxed{\boldsymbol{\nabla}t = (\cos\theta + \sin\theta\cos\phi)\hat{\mathbf{r}} + (-\sin\theta + \cos\theta\cos\phi)\hat{\boldsymbol{\theta}} + \frac{1}{\cancel{\sin\theta}}(-\cancel{\sin\theta}\sin\phi)\hat{\boldsymbol{\phi}}}$

$$\begin{aligned}\nabla^2 t &= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}t)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(\cos\theta + \sin\theta\cos\phi)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta(-\sin\theta + \cos\theta\cos\phi)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-\sin\phi)\\ &= \frac{1}{r^2}\,2r(\cos\theta + \sin\theta\cos\phi) + \frac{1}{r\sin\theta}(-2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi) - \frac{1}{r\sin\theta}\cos\phi\\ &= \frac{1}{r\sin\theta}[2\sin\theta\cos\theta + 2\sin^2\theta\cos\phi - 2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi - \cos\phi]\\ &= \frac{1}{r\sin\theta}\left[(\sin^2\theta + \cos^2\theta)\cos\phi - \cos\phi\right] = 0.\end{aligned}$$

$\Rightarrow \boxed{\nabla^2 t = 0}$

Check: $r\cos\theta = z,\ r\sin\theta\cos\phi = x \ \Rightarrow$ in Cartesian coordinates $t = x + z$. Obviously Laplacian is zero.

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} t \cdot d\mathbf{l} = t(\mathbf{b}) - t(\mathbf{a})$

Segment 1: $\theta = \frac{\pi}{2},\ \phi = 0,\ r: 0 \to 2.\ d\mathbf{l} = dr\,\hat{\mathbf{r}};\ \boldsymbol{\nabla} t\cdot d\mathbf{l} = (\cos\theta + \sin\theta\cos\phi)dr = (0+1)dr = dr.$

$\int \boldsymbol{\nabla} t\cdot d\mathbf{l} = \int_0^2 dr = 2.$

Segment 2: $\theta = \frac{\pi}{2},\ r = 2,\ \phi: 0 \to \frac{\pi}{2}.\quad d\mathbf{l} = r\sin\theta\, d\phi\,\hat{\boldsymbol{\phi}} = 2\,d\phi\,\hat{\boldsymbol{\phi}}.$

$\boldsymbol{\nabla} t\cdot d\mathbf{l} = (-\sin\phi)(2\,d\phi) = -2\sin\phi\,d\phi.\ \int \boldsymbol{\nabla} t\cdot d\mathbf{l} = -\int_0^{\frac{\pi}{2}} 2\sin\phi\,d\phi = \left. 2\cos\phi\right|_0^{\frac{\pi}{2}} = -2.$

Segment 3: $r = 2,\ \phi = \frac{\pi}{2};\ \theta: \frac{\pi}{2} \to 0.$

$d\mathbf{l} = r\,d\theta\,\hat{\boldsymbol{\theta}} = 2\,d\theta\,\hat{\boldsymbol{\theta}};\ \boldsymbol{\nabla} t\cdot d\mathbf{l} = (-\sin\theta + \cos\theta\cos\phi)(2\,d\theta) = -2\sin\theta\,d\theta.$

$\int \boldsymbol{\nabla} t\cdot d\mathbf{l} = -\int_{\frac{\pi}{2}}^{0} 2\sin\theta\,d\theta = \left. 2\cos\theta\right|_{\frac{\pi}{2}}^{0} = 2.$

Total: $\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} t\cdot d\mathbf{l} = 2 - 2 + 2 = \boxed{2}$. Meanwhile, $t(\mathbf{b}) - t(\mathbf{a}) = [2(1+0)] - [0(\ )] = 2.$ ✓

Problem 1.42 From Fig. 1.42, $\boxed{\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}};\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}};\ \hat{\mathbf{z}} = \hat{\mathbf{z}}}$

Multiply first by $\cos\phi$, second by $\sin\phi$, and subtract:

$\hat{\mathbf{s}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi = \cos^2\phi\,\hat{\mathbf{x}} + \cos\phi\sin\phi\,\hat{\mathbf{y}} + \sin^2\phi\,\hat{\mathbf{x}} - \sin\phi\cos\phi\,\hat{\mathbf{y}} = \hat{\mathbf{x}}(\sin^2\phi + \cos^2\phi) = \hat{\mathbf{x}}.$

So $\boxed{\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}}.}$

Multiply first by $\sin\phi$, second by $\cos\phi$, and add:

$\hat{\mathbf{s}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi = \sin\phi\cos\phi\,\hat{\mathbf{x}} + \sin^2\phi\,\hat{\mathbf{y}} - \sin\phi\cos\phi\,\hat{\mathbf{x}} + \cos^2\phi\,\hat{\mathbf{y}} = \hat{\mathbf{y}}(\sin^2\phi + \cos^2\phi) = \hat{\mathbf{y}}.$

So $\boxed{\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}.}$ $\quad\boxed{\hat{\mathbf{z}} = \hat{\mathbf{z}}.}$

Problem 1.43

(a)
$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\,s(2+\sin^2\phi)\right) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}\,2s(2+\sin^2\phi) + \frac{1}{s}\,s(\cos^2\phi - \sin^2\phi) + 3\\ &= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3\\ &= 4 + \sin^2\phi + \cos^2\phi + 3 = \boxed{8.}\end{aligned}$$

(b) $\int(\boldsymbol{\nabla}\cdot\mathbf{v})d\tau = \int(8)s\,ds\,d\phi\,dz = 8\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi\int_0^5 dz = 8(2)\left(\frac{\pi}{2}\right)(5) = \boxed{40\pi.}$

Meanwhile, the surface integral has five parts:

top: $z = 5,\ d\mathbf{a} = s\,ds\,d\phi\,\hat{\mathbf{z}};\ \mathbf{v}\cdot d\mathbf{a} = 3z\,s\,ds\,d\phi = 15s\,ds\,d\phi.\ \int \mathbf{v}\cdot d\mathbf{a} = 15\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi = 15\pi.$

bottom: $z = 0,\ d\mathbf{a} = -s\,ds\,d\phi\,\hat{\mathbf{z}};\ \mathbf{v}\cdot d\mathbf{a} = -3z\,s\,ds\,d\phi = 0.\ \int \mathbf{v}\cdot d\mathbf{a} = 0.$

back: $\phi = \frac{\pi}{2},\ d\mathbf{a} = ds\,dz\,\hat{\boldsymbol{\phi}};\ \mathbf{v}\cdot d\mathbf{a} = s\sin\phi\cos\phi\,ds\,dz = 0.\ \int \mathbf{v}\cdot d\mathbf{a} = 0.$

left: $\phi = 0,\ d\mathbf{a} = -ds\,dz\,\hat{\boldsymbol{\phi}};\ \mathbf{v}\cdot d\mathbf{a} = -s\sin\phi\cos\phi\,ds\,dz = 0.\ \int \mathbf{v}\cdot d\mathbf{a} = 0.$

front: $s = 2,\ d\mathbf{a} = s\,d\phi\,dz\,\hat{\mathbf{s}};\ \mathbf{v}\cdot d\mathbf{a} = s(2+\sin^2\phi)s\,d\phi\,dz = 4(2+\sin^2\phi)d\phi\,dz.$

$\int \mathbf{v}\cdot d\mathbf{a} = 4\int_0^{\frac{\pi}{2}}(2+\sin^2\phi)d\phi\int_0^5 dz = (4)(\pi + \frac{\pi}{4})(5) = 25\pi.$

So $\oint \mathbf{v}\cdot d\mathbf{a} = 15\pi + 25\pi = 40\pi.$ ✓

(c)
$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{v} &= \left(\frac{1}{s}\frac{\partial}{\partial\phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right)\hat{\mathbf{s}} + \left(\frac{\partial}{\partial z}\left(s(2+\sin^2\phi)\right) - \frac{\partial}{\partial s}(3z)\right)\hat{\boldsymbol{\phi}}\\ &\quad + \frac{1}{s}\left(\frac{\partial}{\partial s}(s^2\sin\phi\cos\phi) - \frac{\partial}{\partial\phi}\left(s(2+\sin^2\phi)\right)\right)\hat{\mathbf{z}}\\ &= \frac{1}{s}(2s\sin\phi\cos\phi - s\,2\sin\phi\cos\phi)\,\hat{\mathbf{z}} = \boxed{\mathbf{0}.}\end{aligned}$$

Problem 1.44

(a) $3(3^2) - 2(3) - 1 = 27 - 6 - 1 = \boxed{20.}$

(b) $\cos\pi = \boxed{-1.}$

(c) $\boxed{\text{zero.}}$

(d) $\ln(-2+3) = \ln 1 = \boxed{\text{zero.}}$

Problem 1.45

(a) $\int_{-2}^{2}(2x+3)\frac{1}{3}\delta(x)\,dx = \frac{1}{3}(0+3) = \boxed{1.}$

(b) By Eq. 1.94, $\delta(1-x) = \delta(x-1)$, so $1 + 3 + 2 = \boxed{6.}$

(c) $\int_{-1}^{1} 9x^2\,\frac{1}{3}\,\delta(x+\frac{1}{3})\,dx = 9\left(-\frac{1}{3}\right)^2\frac{1}{3} = \boxed{\frac{1}{3}.}$

(d) $\boxed{1 \text{ (if } a > b\text{), } 0 \text{ (if } a < b\text{).}}$

Problem 1.46

(a) $\int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}\delta(x)\right]dx = x\,f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left(x\,f(x)\right)\delta(x)\,dx.$

The first term is zero, since $\delta(x) = 0$ at $\pm\infty$; $\frac{d}{dx}\left(x\,f(x)\right) = x\frac{df}{dx} + \frac{dx}{dx}f = x\frac{df}{dx} + f.$

So the integral is $-\int_{-\infty}^{\infty}\left(x\frac{df}{dx} + f\right)\delta(x)\,dx = 0 - f(0) = -f(0) = -\int_{-\infty}^{\infty} f(x)\delta(x)\,dx.$

So, $x\frac{d}{dx}\delta(x) = -\delta(x).$ qed

(b) $\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f(x)\theta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\theta(x)dx = f(\infty) - \int_0^{\infty}\frac{df}{dx}dx = f(\infty) - \left(f(\infty) - f(0)\right)$
$= f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx$. So $\frac{d\theta}{dx} = \delta(x).$ qed

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Problem 1.47

(a) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}').}$ Check: $\int\rho(\mathbf{r})d\tau = q\int\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau = q.$ ✓

(b) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{a}) - q\delta^3(\mathbf{r}).}$

(c) Evidently $\rho(r) = A\delta(r-R)$. To determine the constant A, we require

$Q = \int\rho\,d\tau = \int A\delta(r-R)4\pi r^2\,dr = A\,4\pi R^2.$ So $A = \frac{Q}{4\pi R^2}.$ $\boxed{\rho(r) = \frac{Q}{4\pi R^2}\delta(r-R).}$

Problem 1.48

(a) $a^2 + \mathbf{a}\cdot\mathbf{a} + a^2 = \boxed{3a^2.}$

(b) $\int(\mathbf{r}-\mathbf{b})^2\frac{1}{5^3}\delta^3(\mathbf{r})\,d\tau = \frac{1}{125}b^2 = \frac{1}{125}(4^2+3^2) = \boxed{\frac{1}{5}.}$

(c) $c^2 = 25 + 9 + 4 = 38 > 36 = 6^2$, so $\mathbf{c}$ is outside $\mathcal{V}$, so the integral is $\boxed{\text{zero.}}$

(d) $\left(\mathbf{e} - (2\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}})\right)^2 = \left(1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + (-1)\,\hat{\mathbf{z}}\right)^2 = 1 + 1 = 2 < (1.5)^2 = 2.25$, so $\mathbf{e}$ is inside $\mathcal{V}$,
and hence the integral is $\mathbf{e}\cdot(\mathbf{d}-\mathbf{e}) = (3,2,1)\cdot(-2,0,2) = -6 + 0 + 2 = \boxed{-4.}$

Problem 1.49

First method: use Eq. 1.99 to write $J = \int e^{-r}\left(4\pi\boldsymbol{\delta}^3(\mathbf{r})\right)\,d\tau = 4\pi e^{-0} = \boxed{4\pi.}$
Second method: integrating by parts (use Eq. 1.59).

$$J = -\int_{\mathcal{V}} \frac{\hat{\mathbf{r}}}{r^2} \cdot \boldsymbol{\nabla}(e^{-r})\, d\tau + \oint_{\mathcal{S}} e^{-r}\frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a}. \quad \text{But} \quad \boldsymbol{\nabla}\left(e^{-r}\right) = \left(\frac{\partial}{\partial r}e^{-r}\right)\hat{\mathbf{r}} = -e^{-r}\hat{\mathbf{r}}.$$

$$= \int \frac{1}{r^2}e^{-r}4\pi r^2\, dr + \int e^{-r}\frac{\hat{\mathbf{r}}}{r^2} \cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}} = 4\pi \int_0^R e^{-r}\, dr + e^{-R}\int \sin\theta\, d\theta\, d\phi$$

$$= 4\pi\left(-e^{-r}\right)\Big|_0^R + 4\pi e^{-R} = 4\pi\left(-e^{-R} + e^{-0}\right) + 4\pi e^{-R} = 4\pi. \checkmark \quad (\text{Here } R = \infty,\ \text{so } e^{-R} = 0.)$$

Problem 1.50 (a) $\boldsymbol{\nabla}\cdot\mathbf{F_1} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(x^2\right) = \boxed{0}$; $\quad \boldsymbol{\nabla}\cdot\mathbf{F_2} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = \boxed{3}$

$$\boldsymbol{\nabla}\times\mathbf{F_1} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = -\hat{\mathbf{y}}\frac{\partial}{\partial x}\left(x^2\right) = \boxed{-2x\hat{\mathbf{y}}}; \quad \boldsymbol{\nabla}\times\mathbf{F_2} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \boxed{\mathbf{0}}$$

$\boxed{\mathbf{F_2} \text{ is a gradient; } \mathbf{F_1} \text{ is a curl}}$ $\quad \boxed{U_2 = \frac{1}{2}\left(x^3 + y^2 + z^2\right)}$ would do ($\mathbf{F_2} = \boldsymbol{\nabla}U_2$).
For $\mathbf{A_1}$, we want $\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = 0$; $\quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$. $\quad A_y = \frac{x^3}{3}$, $A_x = A_z = 0$ would do it.
$\boxed{\mathbf{A_1} = \frac{1}{3}x^2\,\hat{\mathbf{y}}}$ $\quad (\mathbf{F_1} = \boldsymbol{\nabla}\times\mathbf{A_1})$. (But these are not unique.)

(b) $\boldsymbol{\nabla}\cdot\mathbf{F_3} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$; $\quad \boldsymbol{\nabla}\times\mathbf{F_3} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{\mathbf{x}}\,(x-x) + \hat{\mathbf{y}}\,(y-y) + \hat{\mathbf{z}}\,(z-z) = \mathbf{0}.$

So $\mathbf{F_3}$ can be written as the gradient of a scalar ($\mathbf{F_3} = \boldsymbol{\nabla}U_3$) and as the curl of a vector ($\mathbf{F_3} = \boldsymbol{\nabla}\times\mathbf{A_3}$). In fact, $\boxed{U_3 = xyz}$ does the job. For the vector potential, we have

$$\left\{\begin{array}{lll} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, & \text{which suggests} & A_z = \frac{1}{4}y^2z + f(x,z);\ A_y = -\frac{1}{4}yz^2 + g(x,y) \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz, & \text{suggesting} & A_x = \frac{1}{4}z^2x + h(x,y);\ A_z = -\frac{1}{4}zx^2 + j(y,z) \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy, & \text{so} & A_y = \frac{1}{4}x^2y + k(y,z);\ A_x = -\frac{1}{4}xy^2 + l(x,z) \end{array}\right\}$$

Putting this all together: $\boxed{\mathbf{A_3} = \frac{1}{4}\left\{x\left(z^2-y^2\right)\hat{\mathbf{x}} + y\left(x^2-z^2\right)\hat{\mathbf{y}} + z\left(y^2-x^2\right)\hat{\mathbf{z}}\right\}}$ (again, not unique).

Problem 1.51
(d) $\Rightarrow$ (a): $\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\times(-\boldsymbol{\nabla}U) = \mathbf{0}$ $\quad$ (Eq. 1.44 – curl of gradient is always zero).
(a) $\Rightarrow$ (c): $\oint \mathbf{F}\cdot d\mathbf{l} = \int(\boldsymbol{\nabla}\times\mathbf{F})\cdot d\mathbf{a} = 0$ (Eq. 1.57–Stokes' theorem).
(c) $\Rightarrow$ (b): $\int_{\mathbf{a}\ I}^{\mathbf{b}}\mathbf{F}\cdot d\mathbf{l} - \int_{\mathbf{a}\ II}^{\mathbf{b}}\mathbf{F}\cdot d\mathbf{l} = \int_{\mathbf{a}\ I}^{\mathbf{b}}\mathbf{F}\cdot d\mathbf{l} + \int_{\mathbf{b}\ II}^{\mathbf{a}}\mathbf{F}\cdot d\mathbf{l} = \oint\mathbf{F}\cdot d\mathbf{l} = 0$, so

$$\int_{\mathbf{a}\ I}^{\mathbf{b}}\mathbf{F}\cdot d\mathbf{l} = \int_{\mathbf{a}\ II}^{\mathbf{b}}\mathbf{F}\cdot d\mathbf{l}.$$

(b) $\Rightarrow$ (c): same as (c) $\Rightarrow$ (b), only in reverse; (c) $\Rightarrow$ (a): same as (a)$\Rightarrow$ (c).

Problem 1.52
(d) $\Rightarrow$ (a): $\boldsymbol{\nabla}\cdot\mathbf{F} = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{W}) = 0$ $\quad$ (Eq 1.46—divergence of curl is always zero).
(a) $\Rightarrow$ (c): $\oint\mathbf{F}\cdot d\mathbf{a} = \int(\boldsymbol{\nabla}\cdot\mathbf{F})\,d\tau = 0$ (Eq. 1.56—divergence theorem).

(c) $\Rightarrow$ (b): $\int_I \mathbf{F}\cdot d\mathbf{a} - \int_{II} \mathbf{F}\cdot d\mathbf{a} = \oint \mathbf{F}\cdot d\mathbf{a} = 0$, so

$$\int_I \mathbf{F}\cdot d\mathbf{a} = \int_{II} \mathbf{F}\cdot d\mathbf{a}.$$

(*Note:* sign change because for $\oint \mathbf{F}\cdot d\mathbf{a}$, d**a** is *outward*, whereas for surface II it is *inward*.)
(b) $\Rightarrow$ (c): same as (c) $\Rightarrow$ (b), in reverse; (c)$\Rightarrow$ (a): same as (a)$\Rightarrow$ (c) .

Problem 1.53

In Prob. 1.15 we found that $\boldsymbol{\nabla}\cdot\mathbf{v}_a = 0$; in Prob. 1.18 we found that $\boldsymbol{\nabla}\times\mathbf{v}_c = \mathbf{0}$. So

$\boxed{\mathbf{v}_c \text{ can be written as the gradient of a scalar; } \mathbf{v}_a \text{ can be written as the curl of a vector.}}$

(a) To find t:

(1) $\frac{\partial t}{\partial x} = y^2 \Rightarrow t = y^2 x + f(y,z)$

(2) $\frac{\partial t}{\partial y} = \left(2xy + z^2\right)$

(3) $\frac{\partial t}{\partial z} = 2yz$

From (1) & (3) we get $\frac{\partial f}{\partial z} = 2yz \Rightarrow f = yz^2 + g(y) \Rightarrow t = y^2 x + yz^2 + g(y)$, so $\frac{\partial t}{\partial y} = 2xy + z^2 + \frac{\partial g}{\partial y} = 2xy + z^2$ (from (2)) $\Rightarrow \frac{\partial g}{\partial y} = 0$. We may as well pick $g = 0$; then $\boxed{t = xy^2 + yz^2.}$

(b) To find $\mathbf{W}$: $\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = x^2$; $\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} = 3z^2 x$; $\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} = -2xz$.
Pick $W_x = 0$; then

$$\begin{aligned} \frac{\partial W_z}{\partial x} &= -3xz^2 \Rightarrow W_z = -\frac{3}{2}x^2 z^2 + f(y,z) \\ \frac{\partial W_y}{\partial x} &= -2xz \Rightarrow W_y = -x^2 z + g(y,z). \end{aligned}$$

$\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = \frac{\partial f}{\partial y} + x^2 - \frac{\partial g}{\partial z} = x^2 \Rightarrow \frac{\partial f}{\partial y} - \frac{\partial g}{\partial z} = 0$. May as well pick $f = g = 0$.

$\boxed{\mathbf{W} = -x^2 z\,\hat{\mathbf{y}} - \tfrac{3}{2}x^2 z^2\,\hat{\mathbf{z}}.}$

Check: $$\boldsymbol{\nabla}\times\mathbf{W} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x^2 z & -\frac{3}{2}x^2 z^2 \end{vmatrix} = \hat{\mathbf{x}}\left(x^2\right) + \hat{\mathbf{y}}\left(3xz^2\right) + \hat{\mathbf{z}}\left(-2xz\right).\checkmark$$

You can add any gradient ($\boldsymbol{\nabla} t$) to $\mathbf{W}$ without changing its curl, so this answer is far from unique. Some other solutions:

$\mathbf{W} = xz^3\,\hat{\mathbf{x}} - x^2 z\,\hat{\mathbf{y}}$;

$\mathbf{W} = \left(2xyz + xz^3\right)\hat{\mathbf{x}} + x^2 y\,\hat{\mathbf{z}}$;

$\mathbf{W} = xyz\,\hat{\mathbf{x}} - \frac{1}{2}x^2 z\,\hat{\mathbf{y}} + \frac{1}{2}x^2\left(y - 3z^2\right)\hat{\mathbf{z}}$.

Problem 1.54

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, r^2\cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right) \\
&= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right) \\
&= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.
\end{aligned}$$

$$\begin{aligned}
\int(\boldsymbol{\nabla}\cdot\mathbf{v})\,d\tau &= \int(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi \\
&= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}.}
\end{aligned}$$

Surface consists of four parts:
(1) *Curved:* $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}};\ r = R.\quad \mathbf{v}\cdot d\mathbf{a} = \left(R^2\cos\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right).$

$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi = R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4}.$$

(2) *Left:* $d\mathbf{a} = -r\,dr\,d\theta\,\hat{\boldsymbol{\phi}};\ \phi = 0.\quad \mathbf{v}\cdot d\mathbf{a} = \left(r^2\cos\theta\sin\phi\right)(r\,dr\,d\theta) = 0.\quad \int\mathbf{v}\cdot d\mathbf{a} = 0.$
(3) *Back:* $d\mathbf{a} = r\,dr\,d\theta\,\hat{\boldsymbol{\phi}};\ \phi = \pi/2.\quad \mathbf{v}\cdot d\mathbf{a} = \left(-r^2\cos\theta\sin\phi\right)(r\,dr\,d\theta) = -r^3\cos\theta\,dr\,d\theta.$

$$\int\mathbf{v}\cdot d\mathbf{a} = \int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\,d\theta = -\left(\frac{1}{4}R^4\right)(+1) = -\frac{1}{4}R^4.$$

(4) *Bottom:* $d\mathbf{a} = r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}};\ \theta = \pi/2.\quad \mathbf{v}\cdot d\mathbf{a} = \left(r^2\cos\phi\right)(r\,dr\,d\phi).$

$$\int\mathbf{v}\cdot d\mathbf{a} = \int_0^R r^3\,dr\int_0^{\pi/2}\cos\phi\,d\phi = \frac{1}{4}R^4.$$

Total: $\oint\mathbf{v}\cdot d\mathbf{a} = \pi R^4/4 + 0 - \frac{1}{4}R^4 + \frac{1}{4}R^4 = \frac{\pi R^4}{4}.\quad\checkmark$

Problem 1.55

$$\boldsymbol{\nabla}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0\end{vmatrix} = \hat{\mathbf{z}}\,(b-a).\quad\text{So}\quad \int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = (b-a)\pi R^2.$$

$\mathbf{v}\cdot d\mathbf{l} = (ay\,\hat{\mathbf{x}} + bx\,\hat{\mathbf{y}})\cdot(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) = ay\,dx + bx\,dy;\ x^2+y^2 = R^2 \Rightarrow 2x\,dx + 2y\,dy = 0,$
so $dy = -(x/y)\,dx$. So $\mathbf{v}\cdot d\mathbf{l} = ay\,dx + bx(-x/y)\,dx = \frac{1}{y}\left(ay^2 - bx^2\right)\,dx.$

For the "upper" semicircle, $y = \sqrt{R^2 - x^2}$, so $\mathbf{v} \cdot d\mathbf{l} = \frac{a\left(R^2-x^2\right)-bx^2}{\sqrt{R^2-x^2}}\,dx$.

$$\begin{aligned}\int \mathbf{v} \cdot d\mathbf{l} &= \int\limits_{R}^{-R} \frac{aR^2 - (a+b)x^2}{\sqrt{R^2 - x^2}}\,dx = \left\{aR^2 \sin^{-1}\left(\frac{x}{R}\right) - (a+b)\left[-\frac{x}{2}\sqrt{R^2 - x^2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]\right\}\Bigg|_{+R}^{-R} \\ &= \frac{1}{2}R^2(a-b)\sin^{-1}(x/R)\Big|_{+R}^{-R} = \frac{1}{2}R^2(a-b)\left(\sin^{-1}(-1) - \sin^{-1}(+1)\right) = \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) \\ &= \frac{1}{2}\pi R^2(b-a).\end{aligned}$$

And the same for the lower semicircle (y changes sign, but the limits on the integral are reversed) so $\oint \mathbf{v} \cdot d\mathbf{l} = \pi R^2(b-a)$. ✓

Problem 1.56

(1) $x = z = 0;\ dx = dz = 0;\ y: 0 \to 1.\quad \mathbf{v} \cdot d\mathbf{l} = (yz^2)\,dy = 0;\quad \int \mathbf{v} \cdot d\mathbf{l} = 0.$

(2) $x = 0;\ z = 2-2y;\ dz = -2\,dy;\ y: 1 \to 0.\quad \mathbf{v} \cdot d\mathbf{l} = (yz^2)\,dy + (3y+z)\,dz = y(2-2y)^2\,dy - (3y+2-2y)2\,dy;$

$$\int \mathbf{v} \cdot d\mathbf{l} = 2\int\limits_{1}^{0}(2y^3 - 4y^2 + y - 2)\,dy = 2\left(\frac{y^4}{2} - \frac{4y^3}{3} + \frac{y^2}{2} - 2y\right)\Bigg|_1^0 = \frac{14}{3}.$$

(3) $x = y = 0;\ dx = dy = 0;\ z: 2 \to 0.\quad \mathbf{v} \cdot d\mathbf{l} = (3y+z)\,dz = z\,dz;$

$$\int \mathbf{v} \cdot d\mathbf{l} = \int\limits_{2}^{0} z\,dz = \frac{z^2}{2}\Big|_2^0 = -2.$$

Total: $\oint \mathbf{v} \cdot d\mathbf{l} = 0 + \frac{14}{3} - 2 = \boxed{\frac{8}{3}.}$

Meanwhile, Stokes' thereom says $\oint \mathbf{v} \cdot d\mathbf{l} = \int (\boldsymbol{\nabla}\times\mathbf{v}) \cdot d\mathbf{a}$. Here $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, so all we need is $(\boldsymbol{\nabla}\times\mathbf{v})_x = \frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2) = 3 - 2yz.$ Therefore

$$\begin{aligned}\int (\boldsymbol{\nabla}\times\mathbf{v}) \cdot d\mathbf{a} &= \int\int (3 - 2yz)\,dy\,dz = \int_0^1 \left[\int_0^{2-2y}(3-2yz)\,dz\right] dy \\ &= \int_0^1 \left[3(2-2y) - 2y\tfrac{1}{2}(2-2y)^2\right] dy = \int_0^1 (-4y^3 + 8y^2 - 10y + 6)\,dy \\ &= \left(-y^4 + \tfrac{8}{3}y^3 - 5y^2 + 6y\right)\Big|_0^1 = -1 + \tfrac{8}{3} - 5 + 6 = \tfrac{8}{3}. ✓\end{aligned}$$

Problem 1.57

Start at the origin.

(1) $\theta = \frac{\pi}{2},\ \phi = 0;\ r: 0 \to 1.\quad \mathbf{v} \cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = 0.\quad \int \mathbf{v} \cdot d\mathbf{l} = 0.$

(2) $r = 1,\ \theta = \frac{\pi}{2};\ \phi: 0 \to \pi/2.\quad \mathbf{v} \cdot d\mathbf{l} = (3r)(r\sin\theta\,d\phi) = 3\,d\phi.\quad \int \mathbf{v} \cdot d\mathbf{l} = 3\int\limits_0^{\pi/2} d\phi = \frac{3\pi}{2}.$

(3) $\phi = \frac{\pi}{2}$; $r\sin\theta = y = 1$, so $r = \frac{1}{\sin\theta}$, $dr = \frac{-1}{\sin^2\theta}\cos\theta\, d\theta$, $\theta : \frac{\pi}{2} \to \theta_0 \equiv \tan^{-1}(1/2)$.

$$\begin{aligned}\mathbf{v}\cdot d\mathbf{l} &= \left(r\cos^2\theta\right)(dr) - (r\cos\theta\sin\theta)(r\,d\theta) = \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta \\ &= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta = -\frac{\cos\theta}{\sin^3\theta}\,d\theta.\end{aligned}$$

Therefore

$$\int \mathbf{v}\cdot d\mathbf{l} = -\int_{\pi/2}^{\theta_0} \frac{\cos\theta}{\sin^3\theta}\,d\theta = \left.\frac{1}{2\sin^2\theta}\right|_{\pi/2}^{\theta_0} = \frac{1}{2\cdot(1/5)} - \frac{1}{2\cdot(1)} = \frac{5}{2} - \frac{1}{2} = 2.$$

(4) $\theta = \theta_0$, $\phi = \frac{\pi}{2}$; $r : \sqrt{5} \to 0$. $\mathbf{v}\cdot d\mathbf{l} = \left(r\cos^2\theta\right)(dr) = \frac{4}{5}r\,dr$.

$$\int \mathbf{v}\cdot d\mathbf{l} = \frac{4}{5}\int_{\sqrt{5}}^{0} r\,dr = \left.\frac{4}{5}\frac{r^2}{2}\right|_{\sqrt{5}}^{0} = -\frac{4}{5}\cdot\frac{5}{2} = -2.$$

Total:

$$\oint \mathbf{v}\cdot d\mathbf{l} = 0 + \frac{3\pi}{2} + 2 - 2 = \boxed{\tfrac{3\pi}{2}.}$$

Stokes' theorem says this should equal $\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a}$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,3r) - \frac{\partial}{\partial\phi}(-r\sin\theta\cos\theta)\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(r\cos^2\theta\right) - \frac{\partial}{\partial r}(r3r)\right]\hat{\boldsymbol{\theta}} \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(-rr\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}\left(r\cos^2\theta\right)\right]\hat{\boldsymbol{\phi}} \\ &= \frac{1}{r\sin\theta}[3r\cos\theta]\,\hat{\mathbf{r}} + \frac{1}{r}[-6r]\,\hat{\boldsymbol{\theta}} + \frac{1}{r}[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta]\,\hat{\boldsymbol{\phi}} \\ &= 3\cot\theta\,\hat{\mathbf{r}} - 6\,\hat{\boldsymbol{\theta}}.\end{aligned}$$

(1) *Back face:* $d\mathbf{a} = -r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$; $(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 0$. $\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 0$.

(2) *Bottom:* $d\mathbf{a} = -r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 6r\sin\theta\,dr\,d\phi$. $\theta = \frac{\pi}{2}$, so $(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = 6r\,dr\,d\phi$

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = \int_0^1 6r\,dr\int_0^{\pi/2} d\phi = 6\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}. \quad \checkmark$$

Problem 1.58

$\mathbf{v}\cdot d\mathbf{l} = y\,dz$.

(1) *Left side:* $z = a - x$; $dz = -dx$; $y = 0$. Therefore $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(2) *Bottom:* $dz = 0$. Therefore $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(3) *Back:* $z = a - \frac{1}{2}y$; $dz = -1/2\,dy$; $y : 2a \to 0$. $\int \mathbf{v}\cdot d\mathbf{l} = \int_{2a}^{0} y\left(-\frac{1}{2}\,dy\right) = -\frac{1}{2}\frac{y^2}{2}\Big|_{2a}^{0} = \frac{4a^2}{4} = \boxed{a^2.}$

Meanwhile, $\nabla\times\mathbf{v} = \hat{\mathbf{x}}$, so $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$ is the projection of this surface on the $x\,y$ plane $= \frac{1}{2}\cdot a\cdot 2a = a^2$. ✓

Problem 1.59

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, 4r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^2\tan\theta\right)\\ &= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2\left(\cos^2\theta - \sin^2\theta\right) = \frac{4r}{\sin\theta}\left(\sin^2\theta + \cos^2\theta - \sin^2\theta\right)\\ &= 4r\frac{\cos^2\theta}{\sin\theta}.\end{aligned}$$

$$\begin{aligned}\int(\nabla\cdot\mathbf{v})\,d\tau &= \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = \int_0^R 4r^3\,dr\int_0^{\pi/6}\cos^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]\Bigg|_0^{\pi/6}\\ &= 2\pi R^4\left(\frac{\pi}{12}+\frac{\sin 60^\circ}{4}\right) = \frac{\pi R^4}{6}\left(\pi + 3\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right).}\end{aligned}$$

Surface coinsists of two parts:

(1) *The ice cream:* $r = R$; $\phi : 0 \to 2\pi$; $\theta : 0 \to \pi/6$; $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $\mathbf{v}\cdot d\mathbf{a} = \left(R^2\sin\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right) = R^4\sin^2\theta\,d\theta\,d\phi$.

$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/6}\sin^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6} = 2\pi R^4\left(\frac{\pi}{12} - \frac{1}{4}\sin 60^\circ\right) = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right)$$

(2) *The cone:* $\theta = \frac{\pi}{6}$; $\phi : 0 \to 2\pi$; $r : 0 \to R$; $d\mathbf{a} = r\sin\theta\,d\phi\,dr\,\hat{\boldsymbol{\theta}} = \frac{\sqrt{3}}{2}r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\,r^3\,dr\,d\phi$

$$\int\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\int_0^R r^3\,dr\int_0^{2\pi}d\phi = \sqrt{3}\cdot\frac{R^4}{4}\cdot 2\pi = \frac{\sqrt{3}}{2}\pi R^4.$$

Therefore $\int\mathbf{v}\cdot d\mathbf{a} = \frac{\pi R^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3}\right) = \frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right)$. ✓.

Problem 1.60

(a) Corollary 2 says $\oint(\nabla T)\cdot d\mathbf{l} = 0$. Stokes' theorem says $\oint(\nabla T)\cdot d\mathbf{l} = \int[\nabla\times(\nabla T)]\cdot d\mathbf{a}$. So $\int[\nabla\times(\nabla T)]\cdot d\mathbf{a} = 0$, and since this is true for *any* surface, the integrand must vanish: $\nabla\times(\nabla T) = \mathbf{0}$, confirming Eq. 1.44.

(b) Corollary 2 says $\oint(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$. Divergence theorem says $\oint(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int\nabla\cdot(\nabla\times\mathbf{v})\,d\tau$. So $\int\nabla\cdot(\nabla\times\mathbf{v})\,d\tau = 0$, and since this is true for *any* volume, the integrand must vanish: $\nabla(\nabla\times\mathbf{v}) = 0$, confirming Eq. 1.46.

Problem 1.61

(a) Divergence theorem: $\oint\mathbf{v}\cdot d\mathbf{a} = \int(\nabla\cdot\mathbf{v})\,d\tau$. Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant vector. Using product rule #5 in front cover: $\nabla\cdot\mathbf{v} = \nabla\cdot(\mathbf{c}T) = T(\nabla\cdot\mathbf{c}) + \mathbf{c}\cdot(\nabla T)$. But $\mathbf{c}$ is constant so $\nabla\cdot\mathbf{c} = 0$. Therefore we have: $\int\mathbf{c}\cdot(\nabla T)\,d\tau = \int T\mathbf{c}\cdot d\mathbf{a}$. Since $\mathbf{c}$ is constant, take it outside the integrals: $\mathbf{c}\cdot\int\nabla T\,d\tau = \mathbf{c}\cdot\int T\,d\mathbf{a}$. But $\mathbf{c}$

is *any* constant vector—in particular, it could be be $\hat{\mathbf{x}}$, or $\hat{\mathbf{y}}$, or $\hat{\mathbf{z}}$—so each *component* of the integral on left equals corresponding component on the right, and hence

$$\int \boldsymbol{\nabla} T\, d\tau = \int T\, d\mathbf{a}. \qquad \text{qed}$$

(b) Let $\mathbf{v} \to (\mathbf{v}\times\mathbf{c})$ in divergence theorem. Then $\int \boldsymbol{\nabla}\cdot(\mathbf{v}\times\mathbf{c})d\tau = \int(\mathbf{v}\times\mathbf{c})\cdot d\mathbf{a}$. Product rule #6 $\Rightarrow$ $\boldsymbol{\nabla}\cdot(\mathbf{v}\times\mathbf{c}) = \mathbf{c}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) - \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{c}) = \mathbf{c}\cdot(\boldsymbol{\nabla}\times\mathbf{v})$. (Note: $\boldsymbol{\nabla}\times\mathbf{c} = \mathbf{0}$, since $\mathbf{c}$ is constant.) Meanwhile vector indentity (1) says $d\mathbf{a}\cdot(\mathbf{v}\times\mathbf{c}) = \mathbf{c}\cdot(d\mathbf{a}\times\mathbf{v}) = -\mathbf{c}\cdot(\mathbf{v}\times d\mathbf{a})$. Thus $\int \mathbf{c}\cdot(\boldsymbol{\nabla}\times\mathbf{v})\, d\tau = -\int \mathbf{c}\cdot(\mathbf{v}\times d\mathbf{a})$. Take $\mathbf{c}$ outside, and again let $\mathbf{c}$ be $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ then:

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\, d\tau = -\int \mathbf{v}\times d\mathbf{a}. \quad \text{qed}$$

(c) Let $\mathbf{v} = T\boldsymbol{\nabla}U$ in divergence theorem: $\int \boldsymbol{\nabla}\cdot(T\boldsymbol{\nabla}U)\, d\tau = \int T\boldsymbol{\nabla}U\cdot d\mathbf{a}$. Product rule #(5) $\Rightarrow$ $\boldsymbol{\nabla}\cdot(T\boldsymbol{\nabla}U) = T\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}U) + (\boldsymbol{\nabla}U)\cdot(\boldsymbol{\nabla}T) = T\nabla^2 U + (\boldsymbol{\nabla}U)\cdot(\boldsymbol{\nabla}T)$. Therefore

$$\int \left(T\nabla^2 U + (\boldsymbol{\nabla}U)\cdot(\boldsymbol{\nabla}T)\right)\, d\tau = \int(T\boldsymbol{\nabla}U)\cdot d\mathbf{a}. \quad \text{qed}$$

(d) Rewrite (c) with $T \leftrightarrow U$: $\int \left(U\nabla^2 T + (\boldsymbol{\nabla}T)\cdot(\boldsymbol{\nabla}U)\right)\, d\tau = \int(U\boldsymbol{\nabla}T)\cdot d\mathbf{a}$. Subtract this from (c), noting that the $(\boldsymbol{\nabla}U)\cdot(\boldsymbol{\nabla}T)$ terms cancel:

$$\int \left(T\nabla^2 U - U\nabla^2 T\right)\, d\tau = \int(T\boldsymbol{\nabla}U - U\boldsymbol{\nabla}T)\cdot d\mathbf{a}. \quad \text{qed}$$

(e) Stokes' theorem: $\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = \oint \mathbf{v}\cdot d\mathbf{l}$. Let $\mathbf{v} = \mathbf{c}T$. By Product Rule #(7): $\boldsymbol{\nabla}\times(\mathbf{c}T) = T(\boldsymbol{\nabla}\times\mathbf{c}) - \mathbf{c}\times(\boldsymbol{\nabla}T) = -\mathbf{c}\times(\boldsymbol{\nabla}T)$ (since $\mathbf{c}$ is constant). Therefore, $-\int(\mathbf{c}\times(\boldsymbol{\nabla}T))\cdot d\mathbf{a} = \oint T\mathbf{c}\cdot d\mathbf{l}$. Use vector indentity #1 to rewrite the first term $(\mathbf{c}\times(\boldsymbol{\nabla}T))\cdot d\mathbf{a} = \mathbf{c}\cdot(\boldsymbol{\nabla}T\times d\mathbf{a})$. So $-\int \mathbf{c}\cdot(\boldsymbol{\nabla}T\times d\mathbf{a}) = \oint \mathbf{c}\cdot T\, d\mathbf{l}$. Pull $\mathbf{c}$ outside, and let $\mathbf{c}\to\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ to prove:

$$\int \boldsymbol{\nabla}T\times d\mathbf{a} = -\oint T\, d\mathbf{l}. \quad \text{qed}$$

Problem 1.62

(a) $d\mathbf{a} = R^2\sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}}$. Let the surface be the northern hemisphere. The $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components clearly integrate to zero, and the $\hat{\mathbf{z}}$ component of $\hat{\mathbf{r}}$ is $\cos\theta$, so

$$\mathbf{a} = \int R^2\sin\theta\cos\theta\, d\theta\, d\phi\,\hat{\mathbf{z}} = 2\pi R^2\,\hat{\mathbf{z}}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = 2\pi R^2\,\hat{\mathbf{z}}\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2} = \boxed{\pi R^2\,\hat{\mathbf{z}}.}$$

(b) Let $T = 1$ in Prob. 1.61(a). Then $\boldsymbol{\nabla}T = 0$, so $\oint d\mathbf{a} = 0$. qed

(c) This follows from (b). For suppose $\mathbf{a}_1 \neq \mathbf{a}_2$; then if you put them together to make a closed surface, $\oint d\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2 \neq 0$.

(d) For one such triangle, $d\mathbf{a} = \frac{1}{2}(\mathbf{r}\times d\mathbf{l})$ (since $\mathbf{r}\times d\mathbf{l}$ is the area of the parallelogram, and the direction is perpendicular to the surface), so for the entire conical surface, $\mathbf{a} = \frac{1}{2}\oint \mathbf{r}\times d\mathbf{l}$.

(e) Let $T = \mathbf{c}\cdot\mathbf{r}$, and use product rule #4: $\boldsymbol{\nabla}T = \boldsymbol{\nabla}(\mathbf{c}\cdot\mathbf{r}) = \mathbf{c}\times(\boldsymbol{\nabla}\times\mathbf{r}) + (\mathbf{c}\cdot\boldsymbol{\nabla})\mathbf{r}$. But $\boldsymbol{\nabla}\times\mathbf{r} = 0$, and $(\mathbf{c}\cdot\boldsymbol{\nabla})\mathbf{r} = (c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z})(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = c_x\,\hat{\mathbf{x}} + c_y\,\hat{\mathbf{y}} + c_z\,\hat{\mathbf{z}} = \mathbf{c}$. So Prob. 1.61(e) says

$$\oint T\, d\mathbf{l} = \oint(\mathbf{c}\cdot\mathbf{r})\, d\mathbf{l} = -\int(\boldsymbol{\nabla}T)\times d\mathbf{a} = -\int \mathbf{c}\times d\mathbf{a} = -\mathbf{c}\times\int d\mathbf{a} = -\mathbf{c}\times\mathbf{a} = \mathbf{a}\times\mathbf{c}. \qquad \text{qed}$$

Problem 1.63

(1)

$$\boldsymbol{\nabla\cdot}\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \boxed{\frac{1}{r^2}.}$$

For a sphere of radius R:

$$\left.\begin{aligned} \int \mathbf{v}\cdot d\mathbf{a} &= \int \left(\tfrac{1}{R}\,\hat{\mathbf{r}}\right)\cdot\left(R^2\sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}}\right) = R\int \sin\theta\, d\theta\, d\phi = 4\pi R. \\ \int(\boldsymbol{\nabla\cdot}\mathbf{v})\, d\tau &= \int\left(\tfrac{1}{r^2}\right)\left(r^2\sin\theta\, dr\, d\theta\, d\phi\right) = \left(\int_0^R dr\right)\left(\int \sin\theta\, d\theta\, d\phi\right) = 4\pi R. \end{aligned}\right\} \text{So divergence theorem checks.}$$

Evidently there is *no* delta function at the origin.

$$\boldsymbol{\nabla}\times(r^n\,\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^n\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{n+2}\right) = \frac{1}{r^2}(n+2)r^{n+1} = \boxed{(n+2)r^{n-1}}$$

(except for $n = -2$, for which we already know (Eq. 1.99) that the divergence is $4\pi\boldsymbol{\delta}^3(\mathbf{r})$).

(2) *Geometrically,* it should be zero. Likewise, the curl in the spherical coordinates obviously gives $\boxed{\text{zero.}}$ To be certain there is no lurking delta function here, we integrate over a sphere of radius R, using Prob. 1.61(b): If $\boldsymbol{\nabla}\times(r^n\,\hat{\mathbf{r}}) = \mathbf{0}$, then $\int(\boldsymbol{\nabla}\times\mathbf{v})\, d\tau = \mathbf{0} \stackrel{?}{=} -\oint \mathbf{v}\times d\mathbf{a}$. But $\mathbf{v} = r^n\,\hat{\mathbf{r}}$ and $d\mathbf{a} = R^2\sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}}$ are both in the $\hat{\mathbf{r}}$ directions, so $\mathbf{v}\times d\mathbf{a} = \mathbf{0}$. ✓

Problem 1.64

(a) Since the argument is not a function of angle, Eq. 1.73 says

$$\begin{aligned} D &= -\frac{1}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{2}\right)\frac{2r}{(r^2+\epsilon^2)^{3/2}}\right] = \frac{1}{4\pi r^2}\frac{d}{dr}\left[\frac{r^3}{(r^2+\epsilon^2)^{3/2}}\right] \\ &= \frac{1}{4\pi r^2}\left[\frac{3r^2}{(r^2+\epsilon^2)^{3/2}} - \frac{3}{2}\frac{r^3\, 2r}{(r^2+\epsilon^2)^{5/3}}\right] = \frac{1}{4\pi r^2}\frac{3r^2}{(r^2+\epsilon^2)^{5/2}}\left(r^2+\epsilon^2-r^2\right) = \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}.\ ✓ \end{aligned}$$

(b) Setting $r \to 0$:

$$D(0,\epsilon) = \frac{3\epsilon^2}{4\pi\epsilon^5} = \frac{3}{4\pi\epsilon^3},$$

which goes to infinity as $\epsilon \to 0$. ✓
(c) From (a) it is clear that $D(r,0) = 0$ for $r \neq 0$. ✓
(d)

$$\int D(r,\epsilon)\, 4\pi r^2\, dr = 3\epsilon^2\int_0^\infty \frac{r^2}{(r^2+\epsilon^2)^{5/2}}\, dr = 3\epsilon^2\left(\frac{1}{3\epsilon^2}\right) = 1.\ ✓$$

(I looked up the integral.) Note that (b), (c), and (d) are the defining conditions for $\delta^3(\mathbf{r})$.