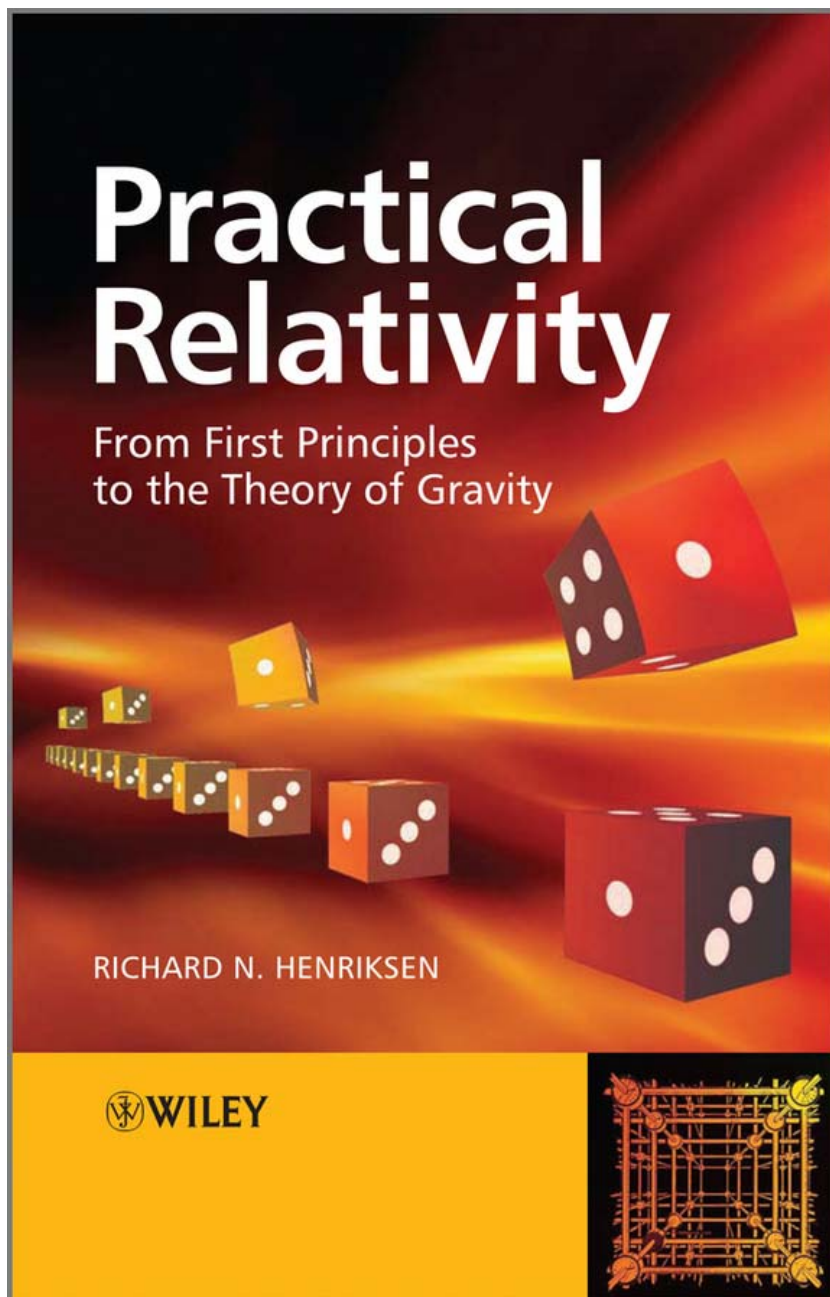


Solutions Manual
for
Practical Relativity
From First Principles to the Theory of
Gravity
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CHAPTER 1

NON-RELATIVITY FOR RELATIVISTS

1.1 The spherical polar base vectors have been parallel-transported to the origin in Figure 1.3.

(a) \hat{e}_x is projected along \hat{e}_r by multiplying first by $\cos \phi$ and then by $\sin \theta$. Similarly, \hat{e}_y is projected along \hat{e}_r by multiplying first by $\sin \phi$ and then by $\sin \theta$. Finally, \hat{e}_z is projected along \hat{e}_r by multiplying by $\cos \theta$. This constructs the first row of the matrix (1.17). Note that \hat{e}_r is indeed a unit vector.

The second row follows by observing that \hat{e}_θ is at an angle θ below the XY plane and the plane defined by \hat{e}_r and \hat{e}_θ intersects the XY plane along the indicated dashed line. Hence \hat{e}_x, \hat{e}_y are resolved along \hat{e}_θ by $\cos \phi \cos \theta$ and $\sin \phi \sin \theta$, respectively, while \hat{e}_z requires the factor, $-\sin \theta$.

The third row simply states that \hat{e}_ϕ is in the XY plane at an angle ϕ to the Y axis.

(b) The transpose of $\underline{\underline{\mathbf{S}}}$ is,

$$\underline{\underline{\mathbf{S}}}^T = \begin{pmatrix} \cos \phi \sin \theta & \sin \phi \sin \theta & \cos \theta \\ \cos \phi \cos \theta & \sin \phi \cos \theta & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix}$$

Hence,

$$\left(\underline{\underline{\tilde{\mathbf{S}}}} \underline{\underline{\mathbf{S}}} \right)_j^i = \tilde{S}_k^i S_j^k = \delta_j^i$$

by direct matrix multiplication. That is,

$$\underline{\underline{\tilde{\mathbf{S}}}} \underline{\underline{\mathbf{S}}} = \underline{\underline{\mathbf{1}}}$$

Consequently, from part (a)

$$\begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{pmatrix} = \underline{\underline{\tilde{\mathbf{S}}}} \begin{pmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{pmatrix}$$

as required. Note that upper indices indicate rows and lower indices indicate columns.

(c) We have in terms of physical components in spherical polars

$$\mathbf{A} = A_{(r)} \hat{\mathbf{e}}_r + A_{(\theta)} \hat{\mathbf{e}}_\theta + A_{(\phi)} \hat{\mathbf{e}}_\phi$$

But

$$\hat{\mathbf{e}}_r = S_i^1 \hat{\mathbf{e}}_{x_i}, \quad \hat{\mathbf{e}}_\theta = S_i^2 \hat{\mathbf{e}}_{x_i}, \quad \hat{\mathbf{e}}_\phi = S_i^3 \hat{\mathbf{e}}_{x_i}$$

Thus

$$\begin{aligned} A_{(r)} &= \hat{\mathbf{e}}_r \cdot \mathbf{A} = S_i^1 \hat{\mathbf{e}}_{x_i} \cdot \mathbf{A} = S_i^1 A_{x_i} \\ A_{(\theta)} &= \hat{\mathbf{e}}_\theta \cdot \mathbf{A} = S_i^2 \hat{\mathbf{e}}_{x_i} \cdot \mathbf{A} = S_i^2 A_{x_i} \\ A_{(\phi)} &= \hat{\mathbf{e}}_\phi \cdot \mathbf{A} = S_i^3 \hat{\mathbf{e}}_{x_i} \cdot \mathbf{A} = S_i^3 A_{x_i} \end{aligned}$$

and so

$$\begin{pmatrix} A_{(r)} \\ A_{(\theta)} \\ A_{(\phi)} \end{pmatrix} = \underline{\underline{\mathbf{S}}} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Evidently $\underline{\underline{\tilde{\mathbf{S}}}}$ gives the inverse result by matrix multiplication as per part (b).

1.2 (a) From Problem 1.1(a) we have, with the usual notation

$$\mathbf{r} = r S_i^1 \hat{\mathbf{e}}_{x_i}$$

Hence by Equation (1.15) and the preceding problem

$$\begin{aligned} \mathbf{e}_r &= S_i^1 \hat{\mathbf{e}}_{x_i} = \hat{\mathbf{e}}_r \\ \mathbf{e}_\theta &= r \frac{\partial S_i^1}{\partial \theta} \hat{\mathbf{e}}_{x_i} = r \hat{\mathbf{e}}_\theta \end{aligned}$$

by direct differentiation and taking $\widehat{\mathbf{e}}_\theta$ from the result of Problem 1.1(a). Similarly

$$\mathbf{e}_\phi = r \frac{\partial S_i^1}{\partial \phi} \widehat{\mathbf{e}}_{x_i} = r \sin \theta \widehat{\mathbf{e}}_\phi$$

Thus

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} \widehat{\mathbf{e}}_r \\ r \widehat{\mathbf{e}}_\theta \\ r \sin \theta \widehat{\mathbf{e}}_\phi \end{pmatrix}$$

which is the same as Equation (1.20).

(b) We see that $d\ell_{(r)} = dr$, $d\ell_{(\theta)} = r d\theta$, and $d\ell_{(\phi)} = r \sin \theta d\phi$. Consequently, by the result of part (a), $\widehat{\mathbf{e}}_\alpha = \frac{\partial \mathbf{r}}{\partial \ell_{(\alpha)}}$.

1.3 We assume a set $\{\mathbf{e}^i\}$ such that $\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i$, and a set $\{\mathbf{n}^i\}$ such that $\mathbf{n}^i \cdot \mathbf{e}_j = \delta_j^i$. Hence by subtraction, $(\mathbf{e}^i - \mathbf{n}^i) \cdot \mathbf{e}_j = 0$ for all i, j . This means that the difference of every pair of reciprocal base vectors is orthogonal to every base vector and therefore must be zero.

1.4 In Equation (1.24) we use the gradient in spherical polars as

$$\nabla = \widehat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\widehat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\widehat{\mathbf{e}}_\phi}{r \sin \phi} \frac{\partial}{\partial \phi}$$

Hence

$$\begin{aligned} \mathbf{e}^r &= \nabla r = \widehat{\mathbf{e}}_r \\ \mathbf{e}^\theta &= \nabla \theta = \frac{\widehat{\mathbf{e}}_\theta}{r} \\ \mathbf{e}^\phi &= \nabla \phi = \frac{\widehat{\mathbf{e}}_\phi}{r \sin \theta} \end{aligned}$$

From Equation (1.22) and Problem 1.2(a)

$$\begin{aligned} \mathbf{e}^r &= \frac{\mathbf{e}_\theta \wedge \mathbf{e}_\phi}{\mathbf{e}_r \cdot (\mathbf{e}_\theta \wedge \mathbf{e}_\phi)} = \frac{r^2 \sin \theta \widehat{\mathbf{e}}_\theta \wedge \widehat{\mathbf{e}}_\phi}{r^2 \sin \theta} = \widehat{\mathbf{e}}_r \\ \mathbf{e}^\theta &= \frac{\mathbf{e}_\phi \wedge \mathbf{e}_r}{r^2 \sin \theta} = \frac{r \sin \theta \widehat{\mathbf{e}}_\phi \wedge \widehat{\mathbf{e}}_r}{r^2 \sin \theta} = \frac{\widehat{\mathbf{e}}_\theta}{r} \\ \mathbf{e}^\phi &= \frac{\mathbf{e}_r \wedge \mathbf{e}_\theta}{r^2 \sin \theta} = \frac{r \widehat{\mathbf{e}}_r \wedge \widehat{\mathbf{e}}_\theta}{r^2 \sin \theta} = \frac{\widehat{\mathbf{e}}_\phi}{r \sin \theta} \end{aligned}$$

1.5 The inverse mapping given acknowledges that x', y', z' are measured along linearly independent directions. Hence by Equation (1.15)

$$\begin{aligned}\mathbf{e}_{x'} &= \frac{\partial \mathbf{r}}{\partial x'} = \hat{\mathbf{e}}_x \\ \mathbf{e}_{y'} &= \frac{\partial \mathbf{r}}{\partial y'} = \hat{\mathbf{e}}_x \cos \beta + \hat{\mathbf{e}}_y \sin \beta \\ \mathbf{e}_{z'} &= \frac{\partial \mathbf{r}}{\partial z'} = \hat{\mathbf{e}}_z \cos \alpha\end{aligned}$$

By Equation (1.24) and the inverse of Equation (1.26), namely

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x - y \cot \beta \\ y \csc \beta \\ z \sec \alpha \end{pmatrix}$$

then

$$\begin{aligned}\mathbf{e}^{x'} &= \hat{\mathbf{e}}_x - \cot \beta \hat{\mathbf{e}}_y \\ \mathbf{e}^{y'} &= \csc \beta \hat{\mathbf{e}}_y \\ \mathbf{e}^{z'} &= \sec \alpha \hat{\mathbf{e}}_z\end{aligned}$$

Hence, $\mathbf{e}^{x'} \cdot \mathbf{e}_{x'} = 1$, $\mathbf{e}^{y'} \cdot \mathbf{e}_{y'} = 1$, $\mathbf{e}^{z'} \cdot \mathbf{e}_{z'} = 1$, and the other three pairs are zero.

1.6 Problem 1.1(c) allows us to write for the Cartesian components of the position vector $\mathbf{r} = r\hat{\mathbf{e}}_r$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \phi \sin \theta \\ r \sin \phi \sin \theta \\ r \cos \theta \end{pmatrix}$$

Hence by (1.30) ($\{1, 2, 3\}$ corresponds to $\{r, \theta, \phi\}$)

$$\begin{aligned}g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = 1 \\ g_{\theta\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2 \\ g_{\phi\phi} &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = r^2 \sin^2 \theta\end{aligned}$$

All other pairs are zero, for example

$$\begin{aligned}g_{r\theta} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} \\ &= r \cos^2 \phi \sin \theta \cos \theta + r \sin^2 \phi \sin \theta \cos \theta - r \sin \theta \cos \theta \\ &= 0\end{aligned}$$

1.7 The line of nodes lies in the XY plane at an angle ϕ to the X axis. Hence the first row of the matrix gives the unit vector \hat{e}_ζ . The axis of κ lies at an angle θ above the axis η , which lies in the XY plane at an angle ϕ to the Y axis. This allows the matrix second row for \hat{e}_κ to be constructed. Finally, $\hat{e}_{Z'}$ forms a plane with the axis η that contains the Z and κ axes. This gives the third row.

1.8 Taking the indicated limit gives respectively

$$\underline{\underline{\mathbf{S}}}_\phi = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{\mathbf{S}}}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

The permutation (3→1→2) of the rows and columns of $\underline{\underline{\mathbf{S}}}_\phi$ gives indeed $\underline{\underline{\mathbf{S}}}_\theta$. This is because the ϕ rotation is about the 3 axis while the θ rotation is about the 1 axis of their respective frames of reference. Moreover, the handedness of the frames is unchanged. The ψ rotation is again about the 3 axis of its systems so that

$$\underline{\underline{\mathbf{S}}}_\psi = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.9 By direct matrix multiplication

$$\begin{aligned} \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi \cos \theta & \cos \phi \cos \theta & \sin \theta \\ \sin \phi \sin \theta & -\sin \theta \cos \phi & \cos \theta \end{pmatrix} \end{aligned}$$

as in Problem 1.7. This shows that the two rotations, $\underline{\underline{\mathbf{S}}}_\phi$, $\underline{\underline{\mathbf{S}}}_\theta$ in that order will convert vectors to the line of node axes.

1.10 Problem (1.9) gives $\underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi$ explicitly, so

$$\begin{aligned} \underline{\underline{\mathbf{S}}} &= \underline{\underline{\mathbf{S}}}_\psi \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi \cos \theta & \cos \phi \cos \theta & \sin \theta \\ \sin \phi \sin \theta & -\sin \theta \cos \phi & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi & \sin \phi \cos \psi + \cos \phi \cos \theta \sin \psi & \sin \theta \sin \psi \\ -\sin \psi \cos \phi - \sin \phi \cos \theta \cos \psi & -\sin \phi \sin \psi + \cos \phi \cos \theta \cos \psi & \sin \theta \cos \psi \\ \sin \phi \sin \theta & -\sin \theta \cos \phi & \cos \theta \end{pmatrix} \end{aligned}$$

1.11 We need only consider $\underline{\underline{\mathbf{S}}}_\phi$ and $\underline{\underline{\mathbf{S}}}_\theta$ since $\underline{\underline{\mathbf{S}}}_\psi$ is the same as $\underline{\underline{\mathbf{S}}}_\phi$ in form. Hence

$$\begin{aligned} \tilde{\underline{\underline{\mathbf{S}}}}_\phi \underline{\underline{\mathbf{S}}}_\phi &= \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

A similar calculation proves the result for $\underline{\underline{\mathbf{S}}}_\theta$.

1.12 Problem 1.9 shows that the line of node unit vectors are given in terms of the unprimed Cartesian base vectors by operating with $\underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi$. Thus the primed Cartesian base vectors will be given by the final rotation $\underline{\underline{\mathbf{S}}}_\psi$, hence

$$\begin{pmatrix} \hat{\mathbf{e}}_{x'} \\ \hat{\mathbf{e}}_{y'} \\ \hat{\mathbf{e}}_{z'} \end{pmatrix} = \underline{\underline{\mathbf{S}}} \begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{pmatrix}$$

Consequently, just as in Problem 1.1(c) we have

$$A'^{(i)} = S_j^i A^{(j)}$$

or $\mathbf{A}' = \underline{\underline{\mathbf{S}}} \mathbf{A}$ in vector notation.

Recall that upper indices indicate column vectors, but that for physical components there is no difference between the elements of column vectors and those of row vectors.

1.13 (a) We can calculate the angular velocity matrix in its defining reference frame from the definition in Equation (1.85). Thus by Problem 1.8

$$\underline{\underline{\Omega}}_\phi = \tilde{\underline{\underline{\mathbf{S}}}}_\phi \dot{\underline{\underline{\mathbf{S}}}}_\phi$$

$$\begin{aligned}
&= \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin \phi & \cos \phi & 0 \\ -\cos \phi & -\sin \phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\phi} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\phi} \equiv \underline{\underline{\sigma}}_z \dot{\phi}
\end{aligned}$$

$\underline{\underline{\Omega}}_\psi$ will have the same form, $\underline{\underline{\Omega}}_\psi = \underline{\underline{\sigma}}_z \dot{\psi}$, and

$$\begin{aligned}
\underline{\underline{\Omega}}_\theta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin \theta & \cos \theta \\ 0 & -\cos \theta & -\sin \theta \end{pmatrix} \dot{\theta} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \dot{\theta} = \underline{\underline{\sigma}}_x \dot{\theta}
\end{aligned}$$

(b) We should have for the velocity of a point at \mathbf{r} , the velocity $\mathbf{v} = -\underline{\underline{\Omega}}\mathbf{r}$ by Equation (1.86), and $\mathbf{v} = \boldsymbol{\omega} \wedge \mathbf{r}$, by Equation (1.110). The first expression gives $\mathbf{v} = -\dot{\phi}y\hat{\mathbf{e}}_x + \dot{\phi}x\hat{\mathbf{e}}_y$ using $\underline{\underline{\Omega}}_\phi$ from part (a) which is equal to $\dot{\phi}\hat{\mathbf{e}}_z \wedge \mathbf{r}$ in inertial axes as required. Similarly using $-\underline{\underline{\Omega}}_\theta\mathbf{r}$, $\mathbf{v} = -\dot{\theta}z\hat{\mathbf{e}}_\kappa + \dot{\theta}\kappa\hat{\mathbf{e}}_z$, which is $\dot{\theta}\hat{\mathbf{e}}_\zeta \wedge \mathbf{r}$ in line of node axes (where $\mathbf{r} = \begin{pmatrix} \zeta \\ \kappa \\ z \end{pmatrix}$).

$\underline{\underline{\Omega}}_\psi$ has the same structure as $\underline{\underline{\Omega}}_\phi$, so that $-\underline{\underline{\Omega}}_\psi\mathbf{r} = -\dot{\psi}y'\hat{\mathbf{e}}_\zeta + \dot{\psi}x'\hat{\mathbf{e}}_{y''}$, which is again $\dot{\psi}\hat{\mathbf{e}}_{z'} \wedge \mathbf{r}$, in the orthonormal set of axes.

1.14 We have that $\underline{\underline{\mathbf{S}}} = \underline{\underline{\mathbf{S}}}_\psi \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi$ and that $\underline{\underline{\Omega}} = \underline{\underline{\tilde{\mathbf{S}}}} \underline{\underline{\dot{\mathbf{S}}}}$. Hence

$$\begin{aligned}
\underline{\underline{\Omega}} &= \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\tilde{\mathbf{S}}}}_\theta \underline{\underline{\tilde{\mathbf{S}}}}_\psi \left(\underline{\underline{\dot{\mathbf{S}}}}_\psi \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi + \underline{\underline{\mathbf{S}}}_\psi \underline{\underline{\dot{\mathbf{S}}}}_\theta \underline{\underline{\mathbf{S}}}_\phi + \underline{\underline{\mathbf{S}}}_\psi \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\dot{\mathbf{S}}}}_\phi \right) \\
&= \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\tilde{\mathbf{S}}}}_\theta \underline{\underline{\Omega}}_\psi \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi + \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\Omega}}_\theta \underline{\underline{\mathbf{S}}}_\phi + \underline{\underline{\Omega}}_\phi
\end{aligned}$$

We recognize from Equation (1.93) that the first term in the last equation transforms $\underline{\underline{\Omega}}_\psi$ to inertial axes, as does the second term for $\underline{\underline{\Omega}}_\theta$. So in inertial axes $\underline{\underline{\Omega}}^I = \underline{\underline{\Omega}}_\psi^I + \underline{\underline{\Omega}}_\theta^I + \underline{\underline{\Omega}}_\phi$ and hence $\boldsymbol{\omega} = \boldsymbol{\omega}_\psi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\phi$ in these axes. To the extent that $\boldsymbol{\omega}$ is a vector, this may be written along any axes. However, the form in terms of the Euler angles can be complicated as shown by our explicit calculation above, which yields the angular dependence. It is often found more readily by geometric resolution.

1.15 We start from

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi$$

Here $\boldsymbol{\omega}_\phi$ is already referred to inertial axes as in the preceding Problem. We obtain $\boldsymbol{\omega}_\theta$ in inertial axes as

$$\underline{\underline{\mathbf{S}}}_\phi \boldsymbol{\omega}_\theta = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$$

or

$$\dot{\theta} \cos \phi \hat{\mathbf{e}}_X + \dot{\theta} \sin \phi \hat{\mathbf{e}}_Y \quad (a)$$

Similarly for $\boldsymbol{\omega}_\psi$ we require

$$\underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{S}}}_\theta \boldsymbol{\omega}_\psi = \begin{pmatrix} \cos \phi & -\sin \phi \cos \theta & \sin \phi \sin \theta \\ \sin \phi & \cos \phi \cos \theta & -\cos \phi \sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

or

$$\dot{\psi} \sin \phi \sin \theta \hat{\mathbf{e}}_X - \dot{\psi} \cos \phi \sin \theta \hat{\mathbf{e}}_Y + \dot{\psi} \cos \theta \hat{\mathbf{e}}_Z \quad (b)$$

Adding $\boldsymbol{\omega}_\phi$ to expressions (a) and (b) yields

$$\begin{aligned} \boldsymbol{\omega} = & (\dot{\phi} + \dot{\psi} \cos \theta) \hat{\mathbf{e}}_Z + (\dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi) \hat{\mathbf{e}}_Y \\ & + (\dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi) \hat{\mathbf{e}}_X \end{aligned}$$

1.16 We know (Problem 1.13) that $\underline{\underline{\mathbf{\Omega}}}_\phi = \dot{\phi} \underline{\underline{\sigma}}_z$. By Equation (1.93) we resolve this along axes rotating about the z axis as

$$\begin{aligned} \underline{\underline{\mathbf{\Omega}}}'' &= \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{\Omega}}}_\phi \underline{\underline{\mathbf{S}}}_\phi = \dot{\phi} \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\sigma}}_z \underline{\underline{\mathbf{S}}}_\phi \\ &= \dot{\phi} \underline{\underline{\mathbf{S}}}_\phi \begin{pmatrix} \sin \phi & -\cos \phi & 0 \\ -\cos \phi & \sin \phi & 0 \\ 0 & 0 & 0 \end{pmatrix} = \dot{\phi} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

However, in the rotating axes $-\dot{\phi} \leftarrow \dot{\phi}$.

1.17 (a) We have from Equation (1.85) $\underline{\underline{\mathbf{\Omega}}} = \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{S}}}_\theta$, hence for the combined rotation

$$\begin{aligned} \underline{\underline{\mathbf{\Omega}}} &= \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{S}}}_\theta (\underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi + \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{S}}}_\theta) \\ &= \underline{\underline{\mathbf{\Omega}}}_\phi + \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi \\ &= \underline{\underline{\mathbf{\Omega}}}_\phi + \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{\Omega}}}_\theta \underline{\underline{\mathbf{S}}}_\phi \end{aligned}$$

And the last term is a similarity transformation from line of node axes to inertial axes (inverse of Equation (1.93)).

(b) Explicitly, recalling $\underline{\underline{\mathbf{S}}}_\phi$ and $\underline{\underline{\mathbf{\Omega}}}_\theta$

$$\begin{aligned}\underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\mathbf{\Omega}}}_\theta \underline{\underline{\mathbf{S}}}_\phi &= \dot{\theta} \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\sigma}}_x \underline{\underline{\mathbf{S}}}_\phi \\ &= \dot{\theta} \underline{\underline{\tilde{\mathbf{S}}}}_\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \sin \phi & -\cos \phi & 0 \end{pmatrix} \\ &= \dot{\theta} \begin{pmatrix} 0 & 0 & -\sin \phi \\ 0 & 0 & \cos \phi \\ \sin \phi & -\cos \phi & 0 \end{pmatrix}\end{aligned}$$

so

$$\begin{aligned}-\dot{\theta} \begin{pmatrix} 0 & 0 & -\sin \phi \\ 0 & 0 & \cos \phi \\ \sin \phi & -\cos \phi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ = \dot{\theta} z \sin \phi \hat{\mathbf{e}}_x - \dot{\theta} z \cos \phi \hat{\mathbf{e}}_y + \dot{\theta} (y \cos \phi - x \sin \phi) \hat{\mathbf{e}}_z\end{aligned}$$

But (see Figure 1.5) we have also $\dot{\theta} \hat{\mathbf{e}}_\zeta \wedge (x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y + z \hat{\mathbf{e}}_z) = -\dot{\theta} x \sin \phi \hat{\mathbf{e}}_x + \dot{\theta} y \cos \phi \hat{\mathbf{e}}_z + \dot{\theta} z \sin \phi \hat{\mathbf{e}}_x - \dot{\theta} z \cos \phi \hat{\mathbf{e}}_y$ as above.

1.18 (a) From Equation (1.93)

$$\underline{\underline{\mathbf{\Omega}}}'' = \underline{\underline{\mathbf{S}}} \underline{\underline{\mathbf{\Omega}}} \underline{\underline{\tilde{\mathbf{S}}}}$$

So in the rotated axes

$$\underline{\underline{\mathbf{\Omega}}}'' = \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi \underline{\underline{\mathbf{\Omega}}} \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\tilde{\mathbf{S}}}}_\theta$$

and using $\underline{\underline{\mathbf{\Omega}}}'$ from Problem 1.17

$$\begin{aligned}\underline{\underline{\mathbf{\Omega}}}'' &= \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{S}}}_\phi \left(\underline{\underline{\mathbf{\Omega}}}_\phi + \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\mathbf{\Omega}}}_\theta \underline{\underline{\mathbf{S}}}_\phi \right) \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\tilde{\mathbf{S}}}}_\theta \\ &= \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\dot{\mathbf{S}}}}_\phi \underline{\underline{\tilde{\mathbf{S}}}}_\phi \underline{\underline{\tilde{\mathbf{S}}}}_\theta + \underline{\underline{\mathbf{S}}}_\theta \underline{\underline{\mathbf{\Omega}}}_\theta \underline{\underline{\tilde{\mathbf{S}}}}_\theta\end{aligned}$$

The first term gives $\underline{\underline{\mathbf{\Omega}}}_\phi$ in the rotated axes and the second term gives $\underline{\underline{\mathbf{\Omega}}}_\theta$ also in these axes.

(b) Recall

$$\underline{\underline{\mathbf{S}}}_\phi = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underline{\underline{\mathbf{S}}}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

The middle pair of the first term give $\underline{\underline{\mathbf{\Omega}}}_\phi$ in the line of node axes as calculated in Problem 1.16, namely

$$\dot{\phi} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the first term becomes in the rotated axes

$$\begin{aligned} \dot{\phi} \underline{\underline{\mathbf{S}}}_{\theta} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{\underline{\mathbf{S}}}_{\theta} &= \dot{\phi} \underline{\underline{\mathbf{S}}}_{\theta} \begin{pmatrix} 0 & \cos \theta & \sin \theta \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \dot{\phi} \begin{pmatrix} 0 & \cos \theta & \sin \theta \\ -\cos \theta & 0 & 0 \\ \sin \theta & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence changing the sign of ϕ in the negative of this operator

$$\begin{aligned} \dot{\phi} \begin{pmatrix} 0 & \cos \theta & \sin \theta \\ -\cos \theta & 0 & 0 \\ \sin \theta & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ \kappa \\ z' \end{pmatrix} \\ = \hat{\mathbf{e}}_{\zeta} \dot{\phi} (\kappa \cos \theta + z' \sin \theta) - \hat{\mathbf{e}}_{\kappa} \dot{\phi} \zeta \cos \theta + \hat{\mathbf{e}}_{z'} \dot{\phi} \zeta \sin \theta \end{aligned}$$

And using Figure 1.5

$$\begin{aligned} \dot{\phi} \hat{\mathbf{e}}_{z'} \wedge (\zeta \hat{\mathbf{e}}_{\zeta} + \kappa \hat{\mathbf{e}}_{\kappa} + z'' \hat{\mathbf{e}}_{z''}) \\ = \dot{\phi} \zeta \cos \theta \hat{\mathbf{e}}_{\kappa} - \dot{\phi} \zeta \sin \theta \hat{\mathbf{e}}_{z''} - \dot{\phi} \kappa \cos \theta \hat{\mathbf{e}}_{\zeta} + \dot{\phi} z'' \sin \theta \hat{\mathbf{e}}_{\zeta} \end{aligned}$$

Changing the sign of ϕ and θ for the (ζ, κ, z'') axes yields the line above (z'' is the same as z').