Solutions Manual

to accompany

STRUCTURAL DYNAMICS
Theory and Applications

Solutions Manual

to accompany

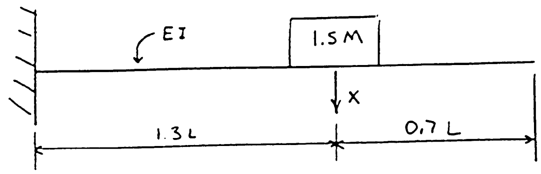
STRUCTURAL DYNAMICS
Theory and Applications

by

Joseph W. Tedesco
Auburn University

Prentice Hall, Upper Saddle River, NJ 07458

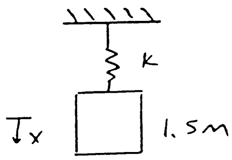
2.1



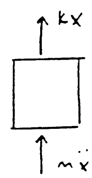
SOLUTION

d'ALEMBERT'S PRINCIPLE

$$\Sigma\,(\text{FORCES})_x - m\ddot{x} = 0$$



FBD



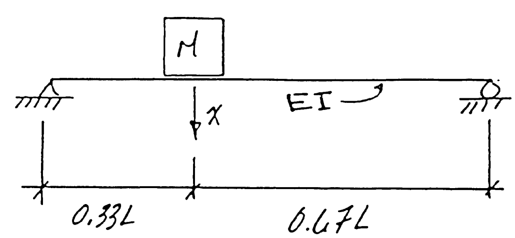
$$kx + m\ddot{x} = 0$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \text{EQUATION OF MOTION}$$

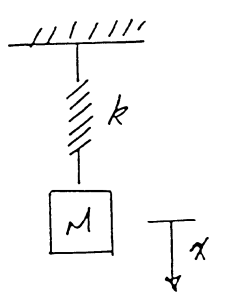
$$k = \frac{3EI}{(1.3L)^3}$$

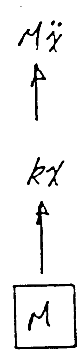
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{1.5M(1.3L)^3}} = 0.954\sqrt{\frac{EI}{ML^3}}$$

2.2



Solution:





2.2 Cont.

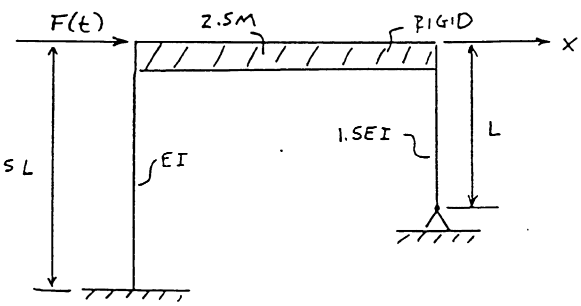
Equation of motion: $M\ddot{x} + kx = 0$ or $\ddot{x} + \frac{k}{M}x = 0$

$$k = \frac{6EIL}{(0.33L)(L-0.33L)[2L(0.33L)-(0.33L)^2-(0.33L)^2]}$$

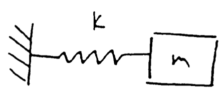
$$k = \frac{61.37EI}{L^3}$$

Natural Frequency: $\omega = \sqrt{\frac{k}{M}} = 7.834\sqrt{\frac{EI}{ML^3}}$

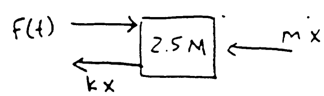
2.3



SOLUTION



FBD



$m = 2.5M$

$\Sigma(\text{FORCES})_x - m\ddot{x} = 0$

$F(t) - kx - m\ddot{x} = 0$

$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$ EQUATION OF MOTION

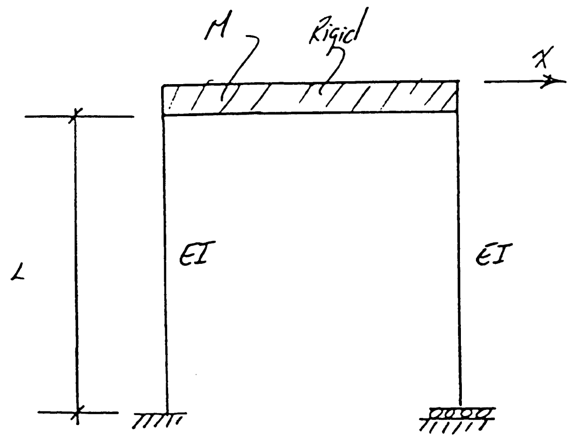
$$k = \frac{12EI}{(1.5L)^3} + \frac{3(1.5EI)}{L^3}$$

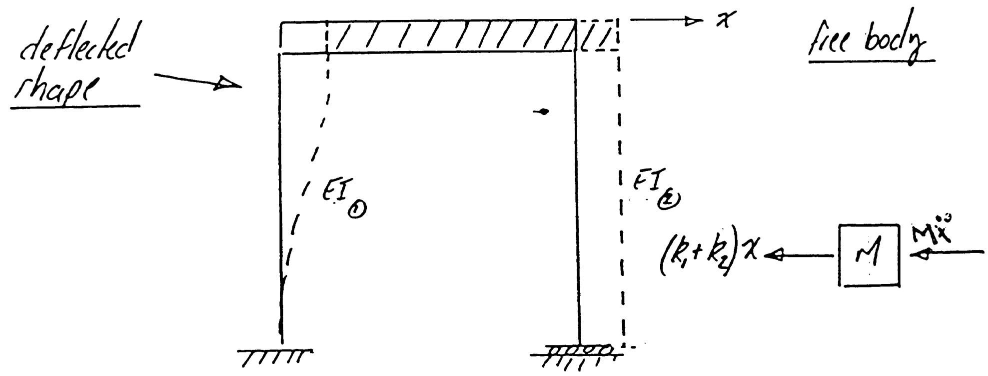
$$= \frac{12(30\times10^6)(150)}{(1.5\times12\times12.0)^3} + \frac{3(1.5)(30\times10^6)(150)}{(12.0\times12)^3}$$

$$= 12{,}140 \text{ LB/IN}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{12{,}140 \text{ LB/IN}}{2.5(1.0 \text{ LB}\cdot\text{sec}^2/\text{IN})}} = 69.7 \text{ rad/sec}$$

2.4



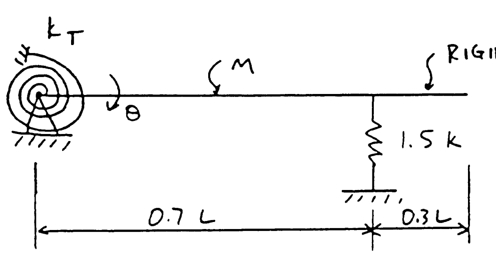


equation of motion: $\boxed{M\ddot{x} + kx = 0 \quad \text{or} \quad \ddot{x} + \frac{k}{M}x = 0}$ ANS

$k_{①} = \frac{12EI}{L^3}$ natural freq. : $\boxed{\omega = \sqrt{\frac{k}{M}} = \left(\frac{12EI}{ML^3}\right)^{1/2}}$ ANS

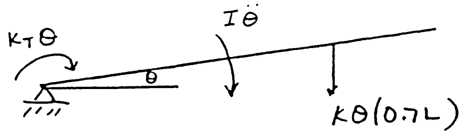
$k_{②} = 0$

2.5



2.5 Cont.

SOLUTION



$\Delta = 0.7L\sin\theta \approx 0.7L\theta$ FOR SMALL θ

$I = \frac{mL^2}{3}$ (ABOUT PIVOT PT)

$$\Sigma M - I\ddot{\theta} = 0$$

$$K_T\theta + k(0.7L)\Delta + I\ddot{\theta} = 0$$

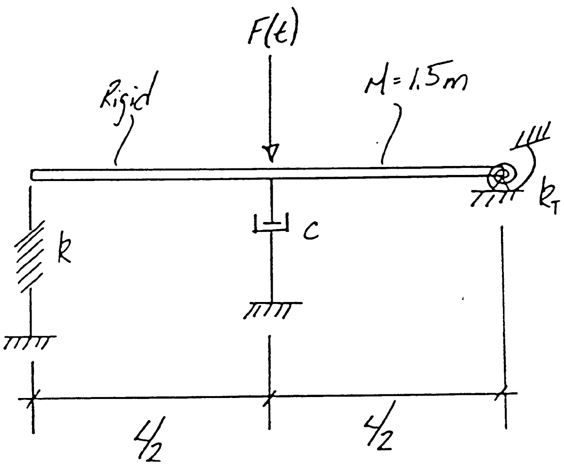
$$k_T\theta + k(0.7L)^2\theta + I\ddot{\theta} = 0$$

$$\frac{mL^2}{3}\ddot{\theta} + k(0.7L)^2\theta + k_T\theta = 0$$

$$\ddot{\theta} + \frac{3\left[k(0.7L)^2 + K_T\right]}{mL^2}\theta = 0$$ EQUATION OF MOTION

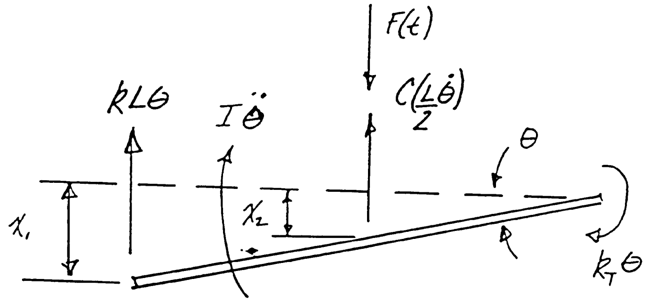
$$\omega = \sqrt{\frac{3k(0.7L)^2 + 3k_T}{mL^2}}$$ NATURAL FREQUENCY

2.6



Solution:

free body



$x_1 = L\sin\theta$ small disp $= L\theta$

$x_2 = \frac{L}{2}\sin\theta$ small disp $= \frac{L}{2}\theta$

$\dot{x}_2 = \frac{L}{2}\dot{\theta}$

2.6 Cont.

$$I = \frac{ML^2}{3} = \frac{1.5mL^2}{3} = \frac{mL^2}{2}$$

equation of motion:

$$k_T\theta + kL\theta(L) + I\ddot{\theta} + c\left(\frac{L}{2}\dot{\theta}\right)\left(\frac{L}{2}\right) = F(t)\left(\frac{L}{2}\right)$$

$$k_T\theta + kL^2\theta + I\ddot{\theta} + \frac{CL^2}{4}\dot{\theta} = F(t)\left(\frac{L}{2}\right)$$

$$I\ddot{\theta} + \frac{CL^2}{4}\dot{\theta} + (kL^2 + k_T)\theta = F(t)\left(\frac{L}{2}\right)$$

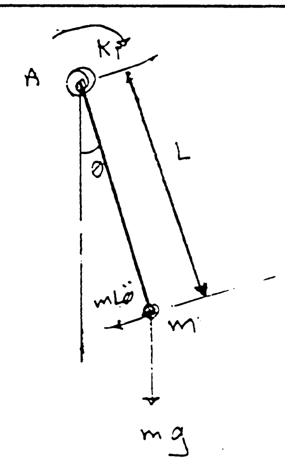
$$\left(\frac{mL^2}{2}\right)\ddot{\theta} + \frac{CL^2}{4}\theta + (kL^2 + k_T)\theta = F(t)\left(\frac{L}{2}\right)$$

$$\boxed{\ddot{\theta} + \frac{C}{2m}\theta + \frac{2(kL^2 + k_T)}{mL^2}\theta = F(t)\left(\frac{1}{mL}\right)}$$ ANS

natural frequency:

$$\boxed{\omega = \sqrt{\frac{2(kL^2 + k_T)}{mL^2}}}$$ ANS

2.7



$$\Sigma M_A = I_A \alpha$$

$$mL\ddot{\theta}(L) + mgL\sin\theta + k_t\theta = 0$$

$$mL^2\ddot{\theta} + (mgL\sin\theta + k_t\theta) = 0$$

2.7 Cont.

For small values of θ

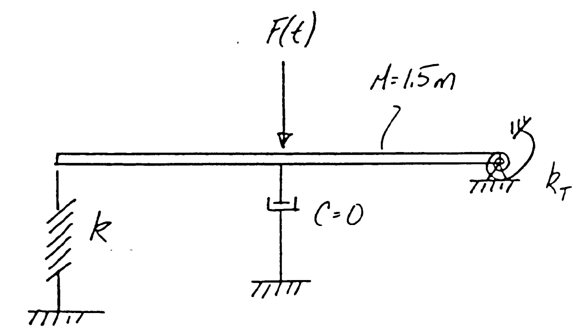
$\sin\theta = \theta$

$$mL^2\ddot{\theta} + (mgL + k_t)\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{k_t}{mL^2}\right)\theta = 0$$

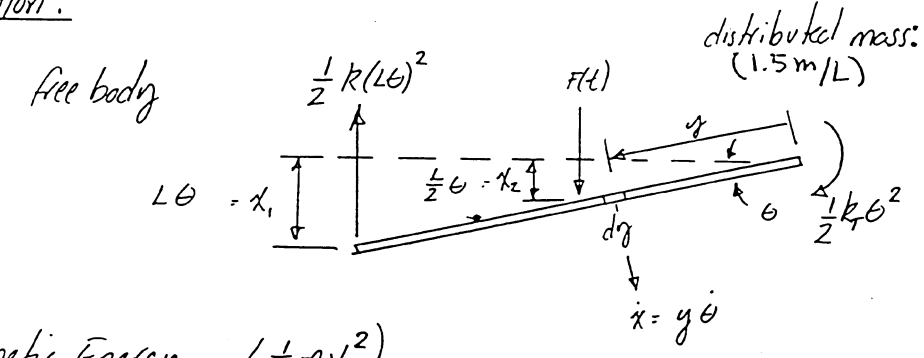
$$\omega = \sqrt{\frac{g}{L} + \frac{k_t}{mL^2}}$$

2.8



Assume a conservative system (i.e. no damping)

Solution:



Kinetic Energy $\left(\frac{1}{2}mv^2\right)$

$$\frac{1}{2}\int_0^L \frac{1.5m}{L}(y\dot{\theta})^2\,dy = \frac{3}{4}\int_0^L \frac{m}{L}\dot{\theta}^2 y^2\,dy = \frac{3}{4}\left(\frac{m}{3L}\dot{\theta}^2 y^3\right)\Big|_0^L$$

$$\boxed{T = \frac{mL^2\dot{\theta}^2}{4}}$$

2.8 Cont.

Potential Energy

$$V = \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k_T\theta^2 - F(t)\left(\frac{L}{2}\right)\theta$$

TOTAL WORK $(T+V) = \text{constant}$

$$\frac{mL^2\dot{\theta}^2}{4} + \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k_T\theta^2 - F(t)\left(\frac{L}{2}\right)\theta = \text{constant}$$

$$\frac{d(T+V)}{d\theta} = 0 = \frac{mL^2}{2}\ddot{\theta}\cancel{\dot{\theta}} + kL^2\cancel{\dot{\theta}}\theta + k_T\cancel{\dot{\theta}}\theta - F(t)\left(\frac{L}{2}\right)\cancel{\dot{\theta}}$$

equation of motion:

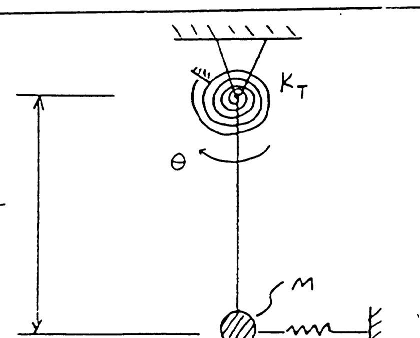
$$\frac{mL^2}{2}\ddot{\theta} + (kL^2 + k_T)\theta = F(t)\left(\frac{L}{2}\right)$$

$$\boxed{\ddot{\theta} + \frac{2(kL^2+k_T)}{mL^2}\theta = F(t)\left(\frac{1}{mL}\right)}$$ ANS

natural frequency:

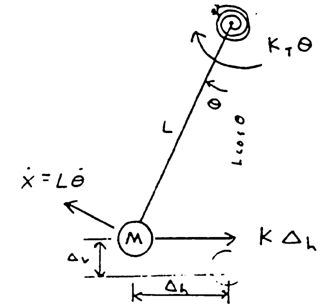
$$\boxed{\omega = \sqrt{\frac{2(kL^2+k_T)}{mL^2}}}$$ ANS

2.9



2.9 Cont.

SOLUTION



$\Delta_h = L\sin\theta \doteq L\theta$

$\Delta_v = L - L\cos\theta = L(1-\cos\theta)$

KINETIC ENERGY

$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(L\dot{\theta})^2$

POTENTIAL ENERGY

$V = mg\Delta_v + \frac{1}{2}K\Delta_h^2 + \frac{1}{2}k_T\theta^2$

$= mgL(1-\cos\theta) + \frac{1}{2}K(L\theta)^2 + \frac{1}{2}k_T\theta^2$

ENERGY METHOD

$T + V = \text{CONSTANT}$

$\frac{d}{dt}(T+V) = 0$

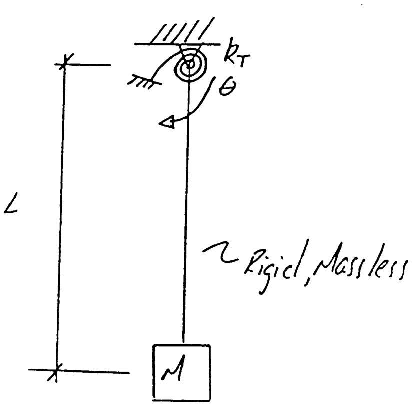
$mL^2\dot{\theta}\ddot{\theta} + mgL(\sin\theta)\dot{\theta} + KL^2\theta\dot{\theta} + k_T\theta\dot{\theta} = 0$

$mL^2\ddot{\theta} + mgL\sin\theta + KL^2\theta + k_T\theta = 0$

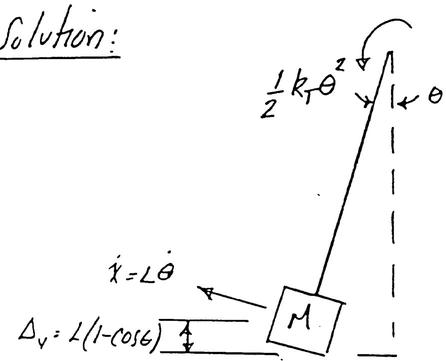
$\ddot{\theta} + \left(\dfrac{MgL + KL^2 + k_T}{ML^2}\right)\theta = 0$ EQUATION OF MOTION

$\omega = \sqrt{\dfrac{MgL + KL^2 + k_T}{ML^2}}$ NATURAL FREQUENCY

2.10



Solution:



2.10 Cont.

Kinetic Energy (T) $\frac{1}{2}mv^2$

$$\frac{1}{2}M(L\dot{\theta})^2$$

Potential energy: (V)

$$MgL(1-\cos\theta) + \frac{1}{2}k_T\theta^2$$

TOTAL WORK: (T+V)

$$\frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}k_T\theta^2 + MgL(1-\cos\theta) = \text{constant}$$

$$\frac{d(T+V)}{d\theta} = 0 = ML^2\ddot{\theta}\cancel{\dot{\theta}} + k_T\cancel{\dot{\theta}}\theta + MgL\sin\theta\,\cancel{\dot{\theta}}$$

$$ML^2\ddot{\theta} + k_T\theta + MgL\theta = 0$$

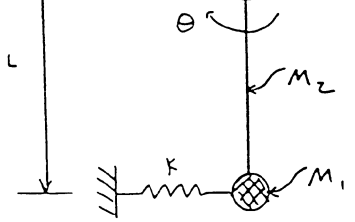
$$ML^2\ddot{\theta} + (k_T + MgL)\theta = 0$$

$$\boxed{\ddot{\theta} + \left(\frac{k_T + MgL}{ML^2}\right)\theta = 0}$$ ANS

natural frequency:

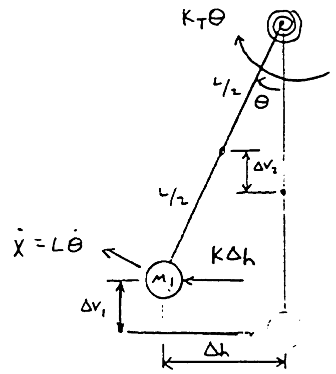
$$\boxed{\omega = \sqrt{\frac{k_T + MgL}{ML^2}}}$$ ANS

2.11



2.11 Cont.

SOLUTION



$$\Delta h = L\sin\theta \approx L\theta$$
$$\Delta v_1 = L(1-\cos\theta)$$
$$\Delta v_2 = \tfrac{L}{2}(1-\cos\theta)$$

$$T = \tfrac{1}{2} M_1 \dot{x}^2 + \tfrac{1}{2} I_0 \dot{\theta}^2$$
$$= \tfrac{1}{2} M_1 (L\dot{\theta})^2 + \tfrac{1}{2}\left(\tfrac{1}{3} M_2 L^2\right)\dot{\theta}^2$$
$$= \tfrac{1}{2} M_1 L^2 \dot{\theta}^2 + \tfrac{1}{6} M_2 L^2 \dot{\theta}^2$$
$$V = M_1 g \Delta v_1 + M_2 g \Delta v_2 + \tfrac{1}{2} k \Delta h^2 + \tfrac{1}{2} k_T \theta^2$$
$$= M_1 g L(1-\cos\theta) + M_2 g \tfrac{L}{2}(1-\cos\theta) + \tfrac{1}{2} k (L\theta)^2 + \tfrac{1}{2} k_T \theta^2$$

ENERGY METHOD

$$T + V = \text{CONSTANT}$$
$$\frac{d}{dt}(T+V) = 0$$

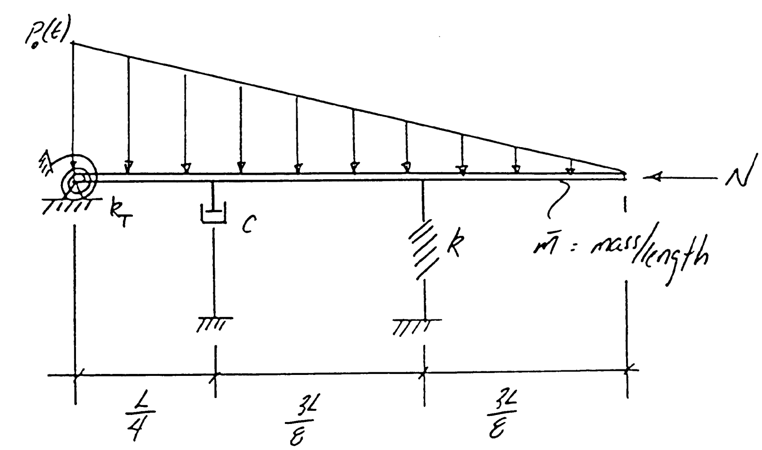
$$M_1 L^2 \dot{\theta}\ddot{\theta} + \tfrac{1}{3} M_2 L^2 \dot{\theta}\ddot{\theta} + M_1 g L(\sin\theta)\dot{\theta} + M_2 g \tfrac{L}{2}(\sin\theta)\dot{\theta} + k L^2 \theta\dot{\theta} + k_T \theta\dot{\theta} = 0$$

$$M_1 L^2 \ddot{\theta} + \tfrac{1}{3} M_2 L^2 \ddot{\theta} + M_1 g L\theta + M_2 g \tfrac{L}{2}\theta + kL^2\theta + K_T\theta = 0$$

$$\ddot{\theta} + \frac{M_1 g L + \tfrac{1}{2} M_2 g L + kL^2 + K_T}{M_1 L^2 + \tfrac{1}{3} M_2 L^2}\,\theta = 0$$ EQUATION OF MOTION

$$\omega = \sqrt{\frac{M_1 g L + \tfrac{1}{2} M_2 g L + kL^2 + k_T}{M_1 L^2 + \tfrac{1}{3} M_2 L^2}}$$ NATURAL FREQUENCY

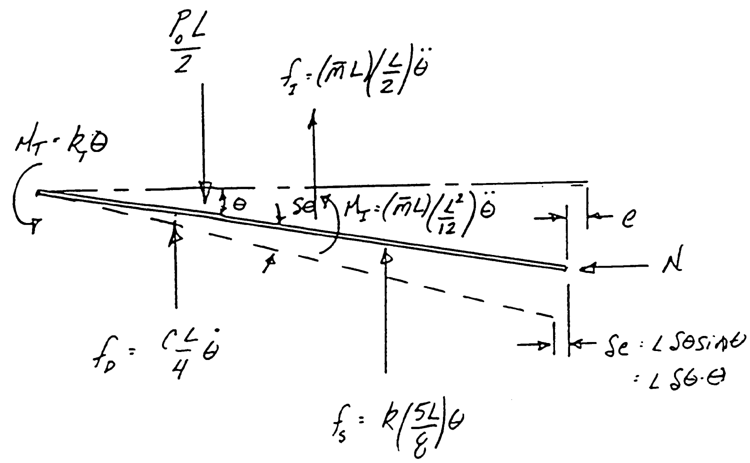
2.12



2.12 Cont.

Solution:

free body



equation of motion:

$$-f_S\left(\frac{5L}{8}\delta\theta\right) - f_D\left(\frac{L}{4}\delta\theta\right) - M_T\,\delta\theta + N\,\delta e + \frac{P_o L}{2}\left(\frac{L}{3}\delta\theta\right)$$

$$\Longrightarrow \quad - f_I\left(\frac{L}{2}\delta\theta\right) - M_I\,\delta\theta = 0$$

$$-\frac{25}{64}L^2 k\theta\cancel{\delta\theta} - c\frac{L^2}{16}\dot{\theta}\cancel{\delta\theta} - k_T\theta\cancel{\delta\theta} + NL\theta\cancel{\delta\theta} + \frac{P_o L^2}{6}\cancel{\delta\theta}$$

$$\Longrightarrow \quad -\frac{\bar{m}L^3}{4}\ddot{\theta}\cancel{\delta\theta} - \frac{\bar{m}L^3}{12}\ddot{\theta}\cancel{\delta\theta} = 0$$

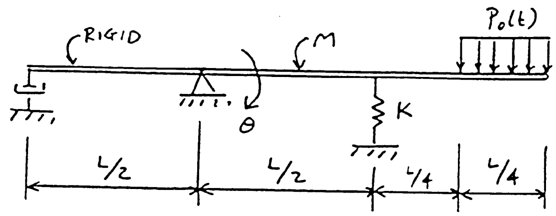
$$\Longrightarrow -\frac{\bar{m}L^3}{3}\ddot{\theta} - \frac{cL^2}{16}\dot{\theta} - \left(\frac{25}{64}kL^2 + k_T - NL\right)\theta + \frac{P_o L^2}{6} = 0$$

$$\boxed{\frac{\bar{m}L^3}{3}\ddot{\theta} + \frac{cL^2}{16}\dot{\theta} + \left(\frac{25}{64}kL^2 + k_T - NL\right)\theta = \frac{P_o L^2}{6}} \quad \text{ANS}$$

Natural frequency: $\omega = \sqrt{\frac{k}{m}}$

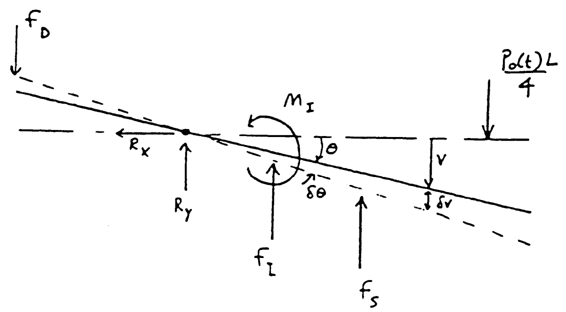
$$\boxed{\omega = \sqrt{\frac{\frac{25}{64}kL^2 + k_T - NL}{\frac{\bar{m}L^3}{3}}}} \quad \text{ANS}$$

2.13



SOLUTION

FBD



FORCES

$$f_s = K\left(\tfrac{L}{2}\right)\theta \qquad f_{P_o} = \frac{P_o L}{4} f(t)$$
$$f_D = c\left(\tfrac{L}{2}\right)\dot{\theta}$$
$$f_I = m\left(\tfrac{L}{4}\right)\ddot{\theta} \qquad M_I = I\ddot{\theta} = \frac{mL^2}{12}\ddot{\theta}$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS

$\delta W = 0$

IN CALCULATING THE VIRTUAL WORK, A QUANTITY IS POSITIVE WHEN THE FORCE ACTS IN THE SAME DIRECTION AS THE VIRTUAL DISPLACEMENT.

$$-f_s\left(\tfrac{L}{2}\delta\theta\right) - f_D\left(\tfrac{L}{2}\delta\theta\right) - f_I\left(\tfrac{L}{4}\delta\theta\right) - M_I(\delta\theta) + f_{P_o}\left(\tfrac{7}{8}L\,\delta\theta\right) = 0$$

$$-K\frac{L^2}{4}\theta\delta\theta - c\frac{L^2}{4}\dot{\theta}\delta\theta - m\frac{L^2}{16}\ddot{\theta}\delta\theta - \frac{mL^2}{12}\ddot{\theta}\delta\theta + \frac{7P_o(t)L^2}{32}\delta\theta = 0$$

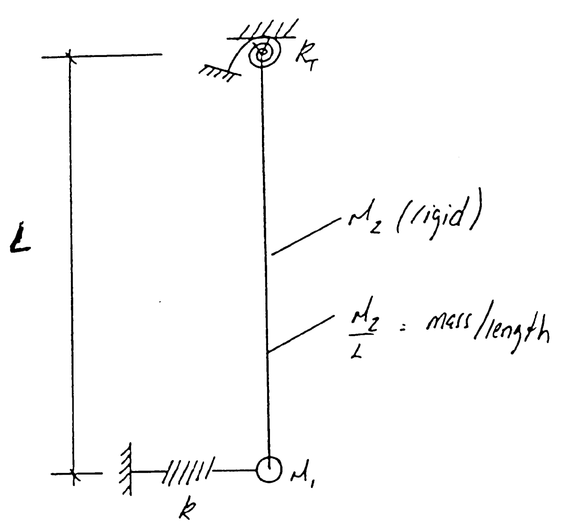
$$\left[\left(\frac{mL^2}{16} + \frac{mL^2}{12}\right)\ddot{\theta} + \left(\frac{cL^2}{4}\right)\dot{\theta} + \left(\frac{KL^2}{4}\right)\theta\right]\delta\theta = \frac{7P_o(t)L^2}{32}\delta\theta$$

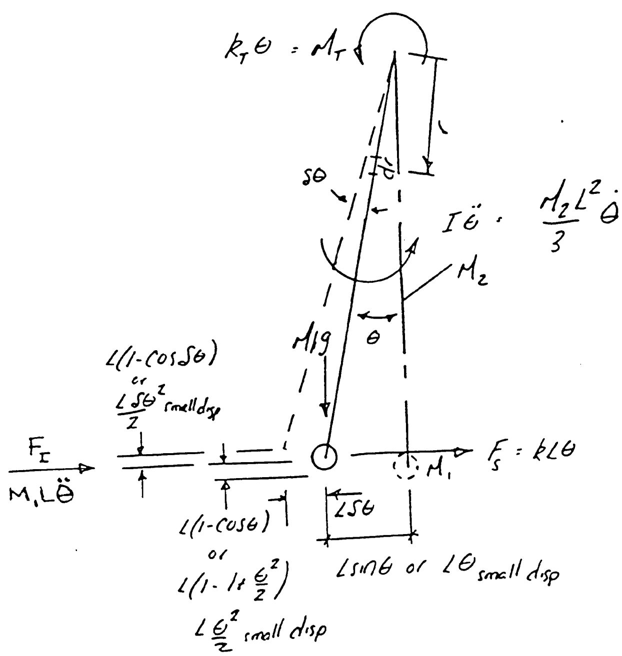
SINCE $\delta\theta \neq 0$

$$\frac{7mL^2}{48}\ddot{\theta} + \frac{cL^2}{4}\dot{\theta} + \frac{KL^2}{4}\theta = \frac{7P_o(t)L^2}{32}$$ EQUATION OF MOTION

$$\omega = \sqrt{\frac{\frac{K\cancel{L^2}}{4}}{\frac{7m\cancel{L^2}}{48}}} = \sqrt{\frac{12K}{7m}}$$ RAD/SEC NATURAL FREQUENCY

2.14





$$-M_T\delta\theta - F_s L\delta\theta - F_I L\delta\theta - I\ddot{\theta}\delta\theta - M_1 g L((1-\cos(\theta+\delta\theta))-(1-\cos\theta)) - \int_0^L \frac{M_2}{L} g\, r\,((1-\cos(\theta+\delta\theta))-(1-\cos\theta))\,dr = 0$$

$$-k_T\theta\delta\theta - kL^2\theta\delta\theta - M_1L^2\ddot{\theta}\delta\theta - \frac{M_2L^2}{3}\ddot{\theta}\delta\theta - M_1 g L\underbrace{((1-\cos(\theta+\delta\theta))-(1-\cos\theta))}_{x} - \int_0^L \frac{M_2}{L} g\, r\underbrace{((1-\cos(\theta+\delta\theta))-(1-\cos\theta))}_{x}dr = 0$$

trigonometric Identities

$$x = (1-\cos(\theta+\delta\theta)) - (1-\cos\theta)$$

$$x = 1-\cos(\theta+\delta\theta) - 1 + \cos\theta$$

$$x = \cos\theta - \cos(\theta+\delta\theta)$$

$$x = \cos\theta - (\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta)$$

$$x = \cos\theta - \cos\theta\cos\delta\theta + \sin\theta\sin\delta\theta$$

small displacements $\cos\theta = 1$, $\cos\delta\theta = 1$, $\sin\theta = \theta$, $\sin\delta\theta = \delta\theta$ so, $x = \theta\delta\theta$

∴

$$k_T\theta\delta\theta + kL^2\theta\delta\theta + M_1L^2\ddot{\theta}\delta\theta + \frac{M_2L^2}{3}\ddot{\theta}\delta\theta + M_1 g L\theta\delta\theta + \underbrace{\int_0^L \frac{M_2}{L} g\, r\,\theta\delta\theta\, dr}_{} = 0$$

$$\frac{M_2 g}{L}\theta\delta\theta \left. \frac{r^2}{2}\right|_0^L = \frac{M_2 g}{L}\theta\delta\theta\left(\frac{L^2}{2}\right) = \frac{M_2 g L}{2}\theta\delta\theta$$

2.14 Cont.

$$k_T\theta\cancel{\delta\theta} + kL^2\theta\cancel{\delta\theta} + M_1L^2\ddot{\theta}\cancel{\delta\theta} + \frac{M_2L^2}{3}\ddot{\theta}\cancel{\delta\theta} + M_1gL\theta\cancel{\delta\theta} + \frac{M_2gL}{2}\theta\cancel{\delta\theta} = 0$$

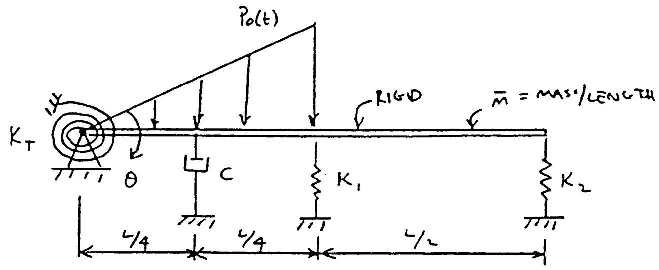
equation of motion:

$$\boxed{\left(\frac{M_2L^2}{3} + M_1L^2\right)\ddot{\theta} + \left(kL^2 + k_T + M_1gL + \frac{M_2gL}{2}\right)\theta = 0} \quad \text{ANS}$$

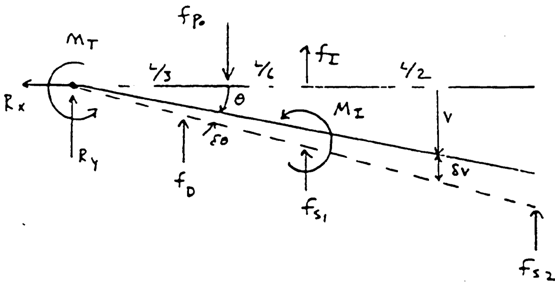
natural frequency:

$$\boxed{\omega = \sqrt{\frac{kL^2 + k_T + M_1gL + \frac{M_2gL}{2}}{\frac{M_2L^2}{3} + M_1L^2}}} \quad \text{ANS}$$

2.15



SOLUTION



PRINCIPLE OF VIRTUAL DISPLACEMENTS

$$\delta W = 0$$

$$-f_{s_1}\left(\tfrac{L}{2}\delta\theta\right) - f_{s_2}(L\delta\theta) - f_D\left(\tfrac{L}{4}\delta\theta\right) - f_I\left(\tfrac{L}{2}\delta\theta\right) - M_I\delta\theta - M_T\delta\theta + f_{P_0}\left(\tfrac{L}{3}\delta\theta\right) = 0$$

FORCES

$$f_{s_1} = K_1\left(\tfrac{L}{2}\theta\right) \qquad M_I = I\ddot{\theta} = \frac{(\bar{m}L)L^2}{12}\ddot{\theta}$$

$$f_{s_2} = k_2(L\theta) \qquad M_T = k_T\theta$$

$$f_D = c\left(\tfrac{L}{4}\dot{\theta}\right) \qquad f_{P_0} = \frac{P_0L}{4}f(t)$$

$$f_I = (\bar{m}L)\left(\tfrac{L}{2}\ddot{\theta}\right)$$

2.15 cont.

SUBSTITUTING INTO VIRTUAL WORK EQUATION:

$$-k_1\frac{L^2}{4}\theta\delta\theta - k_2 L^2\theta\delta\theta - c\frac{L^2}{16}\dot{\theta}\delta\theta - \bar{m}\frac{L^3}{4}\ddot{\theta}\delta\theta - \frac{\bar{m}L^3}{12}\ddot{\theta}\delta\theta - k_T\theta\delta\theta + \frac{P_0(t)L^2}{12}\delta\theta = 0$$

$$\left[\left(\frac{\bar{m}L^3}{4} + \frac{\bar{m}L^3}{12}\right)\ddot{\theta} + \left(\frac{cL^2}{16}\right)\dot{\theta} + \left(\frac{k_1L^2}{4} + k_2L^2 + k_T\right)\theta\right]\delta\theta = \frac{P_0(t)L^2}{12}\delta\theta$$

SINCE $\delta\theta \neq 0$

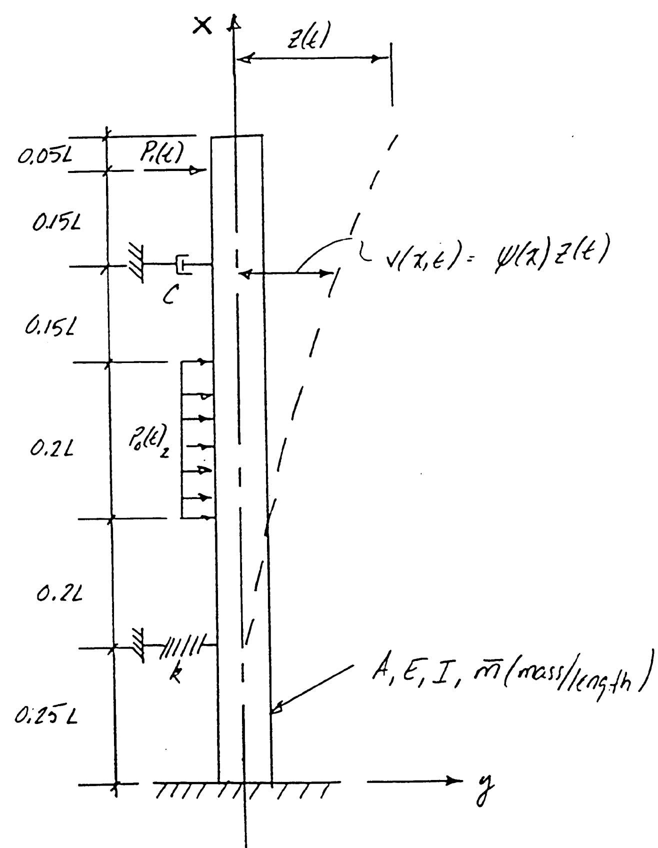
$$\left(\frac{\bar{m}L^3}{3}\right)\ddot{\theta} + \left(\frac{cL^2}{16}\right)\dot{\theta} + \left(\frac{k_1L^2}{4} + k_2L^2 + k_T\right)\theta = \frac{P_0(t)L^2}{12}$$

EQUATION OF MOTION

$$\omega = \sqrt{\frac{\frac{k_1L^2}{4} + k_2L^2 + k_T}{\frac{\bar{m}L^3}{3}}}$$

NATURAL FREQUENCY

2.16



Flexible Column

Assume deflected shape:

$$\psi(x) = \left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right)$$

Use the deflection @ the top of the structure as the generalized coordinate

2.16 Cont.

Solution:

No consentrated mass = 0

$$m^* = \int m(x)\psi^2(x)dx + \sum_i \cancel{M_i\psi^2(x_i)}$$

$$= \bar{m}\int_0^L \left[\left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right)\right]^2 dx$$

$$= \bar{m}\int_0^L \frac{9x^4}{4L^4} - \frac{3x^5}{2L^5} + \frac{x^6}{4L^6}\,dx$$

$$= \bar{m}\left|\frac{9x^5}{20L^4} - \frac{x^6}{4L^5} + \frac{x^7}{28L^6}\right|_0^L$$

$$= \bar{m}\left(\frac{9L^5}{20L^4} - \frac{L^6}{4L^5} + \frac{L^7}{28L^6}\right)$$

$$\boxed{m^* = 0.23571\,\bar{m}L}$$

0 (no distrib. dampers)

$$c^* = \int \cancel{c(x)\psi^2(x)dx} + \sum_i c_i\psi^2(x_i)$$

$$= c\left(\frac{9x_i^4}{4L^4} - \frac{3x_i^5}{2L^5} + \frac{x_i^6}{4L^6}\right) \quad \text{where } x_i = 0.8L$$

$$= c\left(\frac{9(0.8L)^4}{4L^4} - \frac{3(0.8L)^5}{2L^5} + \frac{(0.8L)^6}{4L^6}\right)$$

$$\boxed{c^* = 0.49562\ c}$$

$$k^* = \int \cancel{k(x)\psi^2(x)dx} + \int EI(x)(\psi''(x))^2dx + \sum_i k_i\psi^2(x_i)$$

No distributed spring stiffness = 0

$$\psi(x) = \left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right) = \frac{3x^2}{2L^2} - \frac{x^3}{2L^3}$$

$$\psi'(x) = \frac{3x}{L^2} - \frac{3x^2}{2L^3}$$

$$\psi''(x) = \frac{3}{L^2} - \frac{3x}{L^3}$$

2.16 Cont.

$$k^* = EI\int_0^L \left(\frac{3}{L^2} - \frac{3x}{L^3}\right)^2 dx + k\left(\frac{9x_i^4}{4L^4} - \frac{3x_i^5}{2L^5} + \frac{x_i^6}{4L^6}\right)$$

where $x_i = 0.25L$

$$= EI\int_0^L \left(\frac{9}{L^4} - \frac{18x}{L^5} + \frac{9x^2}{L^6}\right) dx + k\left(\frac{9(0.25L)^4}{4L^4} - \frac{3(0.25L)^5}{2L^5} + \frac{(0.25L)^6}{4L^6}\right)$$

$$= EI\left|\frac{9x}{L^4} - \frac{9x^2}{L^5} + \frac{3x^3}{L^6}\right|_0^L + (0.00739\,k)$$

$$= EI\left(\frac{9L}{L^4} - \frac{9L^2}{L^5} + \frac{3L^3}{L^6}\right) + (0.00314\,k)$$

$$= EI\left(\frac{9}{L^3} - \frac{9}{L^3} + \frac{3}{L^3}\right) + (0.00314\,k)$$

$$\boxed{k^* = \frac{3EI}{L^3} + 0.00739\,k}$$

$$p^*(t) = \int p(x,t)\psi(x)\,dx + \sum_i P_i\,\psi(x_i)$$

$$= P_0\int_{0.45L}^{0.65L} \frac{3x^2}{2L^2} - \frac{x^3}{2L^3}\,dx + P_1\left(\frac{3x_i^2}{2L^2} - \frac{x_i^3}{2L^3}\right)$$

where $x_i = 0.95L$

$$= P_0\left|\frac{x^3}{2L^2} - \frac{x^4}{8L^3}\right|_{0.45L}^{0.65L} + P_1\left(\frac{3(0.95L)^2}{2L^2} - \frac{(0.95L)^3}{2L^3}\right)$$

$$= P_0\left(\frac{(0.65L)^3}{2L^2} - \frac{(0.65L)^4}{8L^3} - \frac{(0.45L)^3}{2L^2} + \frac{(0.45L)^4}{8L^3}\right) + (0.92506\,P_1)$$

$$\boxed{p^*(t) = 0.07456\,P_0 L + 0.92506\,P_1}$$

2.16 Cont.

natural frequency:

$$m^* \ddot{z} + c^* \dot{z} + k^* z = p^*(t)$$

$$(0.23571\, \bar{m} L)\ddot{z} + (0.495\text{62}\, c)\dot{z} + \left(\frac{3EI}{L^3} + 0.00739\, k\right) z$$

equation of motion $\rightarrow$

$$= 0.07456\, P_0 L + 0.92506\, R_0$$

$$\boxed{\omega = \sqrt{\frac{\frac{3EI}{L^3} + 0.00739\, k}{0.23571\, \bar{m} L}}}$$ ANS

find critical Load N_{cr}

$$k^*_{cr} = N_{cr} \int (\psi'(x))^2 dx$$

$$= N_{cr} \int_0^L \left(\frac{3x}{L^2} - \frac{3x^2}{2L^3}\right)^2 dx$$

$$= N_{cr} \int_0^L \left(\frac{9x^2}{L^4} - \frac{9x^3}{L^5} + \frac{9x^4}{4L^6}\right) dx$$

$$= N_{cr} \left| \frac{3x^3}{L^4} - \frac{9x^4}{4L^5} + \frac{9x^5}{20L^6} \right|_0^L$$

$$= N_{cr} \left(\frac{3L^3}{L^4} - \frac{9L^4}{4L^5} + \frac{9L^5}{20L^6}\right)$$

$$\boxed{k^*_{cr} = N_{cr}\left(\frac{1.2}{L}\right) = \frac{1.2\, N_{cr}}{L}}$$ ANS

$$\boxed{k^* - k^*_{cr} = \frac{3EI}{L^3} + 0.00739\, k - \frac{1.2\, N_{cr}}{L} = 0}$$

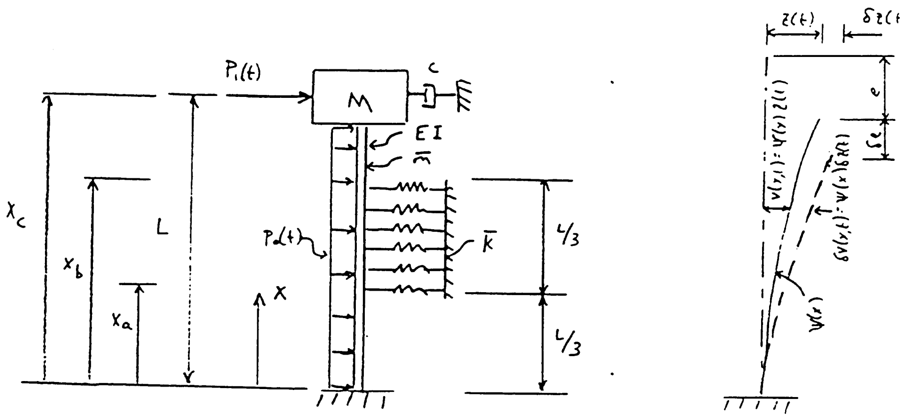
2.16 Cont.

$$\frac{1.2 N_{cr}}{L} = \frac{3EI}{L^3} + 0.00739\,k$$

$$N_{cr} = \left(\frac{3EI}{L^3} + 0.00739\,k\right)\left(\frac{L}{1.2}\right)$$

$$\boxed{N_{cr} = \frac{2.5EI}{L^2} + 0.00616\,kL}$$ ANS

2.17



SOLUTION

CHECK BOUNDARY CONDITIONS

$v(0,t) = v'(0,t) = 0$

$\psi(x) = \frac{3x^2}{L^2} - \frac{x^3}{L^3} \quad \rightarrow \quad \psi(0) = 0$

$\psi'(x) = \frac{6x}{L^2} - \frac{3x^2}{L^3} \quad \rightarrow \quad \psi'(0) = 0$

ALL KINEMATIC BOUNDARY CONDITIONS SATISFIED.

HELPFUL RELATIONSHIPS IN COMPUTING VIRTUAL WORK EXPRESSIONS

$v'(x,t) = \psi'(x)\,z(t) \qquad \delta v(x,t) = \psi(x)\,\delta z(t)$

$v''(x,t) = \psi''(x)\,z(t) \qquad \delta v'(x,t) = \psi'(x)\,\delta z(t)$

$\dot{v}(x,t) = \psi(x)\,\dot{z}(t) \qquad \delta \dot{v}(x,t) = \psi(x)\,\delta \dot{z}(t)$

$\ddot{v}(x,t) = \psi(x)\,\ddot{z}(t) \qquad \delta \ddot{v}(x,t) = \psi(x)\,\delta \ddot{z}(t)$

INERTIA

$$\delta W_{INERTIA} = -\int_0^L \bar{m}\,\ddot{v}(x,t)\,\delta v(x,t)\,dx - M\,\ddot{v}(x_c,t)\,\delta v(x_c,t)$$

$$= -\left[\int_0^L \bar{m}\,\psi(x)^2\,dx + M\,\psi(x_c)^2\right]\ddot{z}\,\delta z$$

DAMPING

$$\delta W_{DAMPING} = -c\,\dot{v}(x_c,t)\,\delta v(x_c,t)$$

$$= -c\,\psi(x_c)^2\,\dot{z}\,\delta z$$

TRANSVERSE LOADS

$$\delta W_p = \int_0^L p_o(x,t)\,\delta v(x,t)\,dx + P_1(t)\,\delta v(x_c,t)$$

$$= \left[\int_0^L p_o(x,t)\,\psi(x)\,dx + P_1(t)\,\psi(x_c)\right]\delta z$$

2.17 Cont.

AXIAL LOAD

$$\delta W_N = N\delta e$$

$$e = \frac{1}{2}\int_0^L (x')^2\,dx \quad \text{AND} \quad \delta e = \frac{1}{2}\int_0 v'\delta v'\,dx$$

$$\delta W_N = \int_0^L N\left[\psi'(x)\right]^2 dx \quad z\delta z$$

SPRINGS

$$\delta W_{SPRING} = -\int_{x_a}^{x_b} \bar{k}\, v(x,t)\,\delta v(x,t)\,dx$$

$$= -\left[\int_{x_a}^{x_b} \bar{k}\,\psi(x)^2\,dx\right] z\delta z$$

BENDING

$$\delta W_{BENDING} = -\int_0^L EI(x)\, v''(x,t)\,\delta v''(x,t)\,dx$$

$$= -\left[\int_0^L EI(x)\left[\psi''(x)\right]^2 dx\right] z\delta z$$

VIRTUAL WORK EQUATION

$$\delta W = 0$$

$$-\left[\int_0^L \bar{m}\,\psi(x)^2 dx + M\psi(x_c)^2\right]\ddot{z}\delta z - \left[c\,\psi(x_c)^2\right]\dot{z}\delta z - \left[\int_{x_a}^{x_b}\bar{k}\,\psi(x)^2 dx + \int_0^L EI(x)\left[\psi''(x)\right]^2 dx\right] z\delta z$$

$$+\left[\int_0^L N\left[\psi'(x)\right]^2 dx\right] z\delta z + \left[\int_0^L p_0(x,t)\,\psi(x)\,dx + P_1(t)\,\psi(x_c)\right]\delta z = 0$$

REARRANGING:

$$\left[\int_0^L \bar{m}\,\psi(x)^2 dx + M\psi(x_c)^2\right]\ddot{z}\delta z + \left[c\,\psi(x_c)^2\right]\dot{z}\delta z + \left[\int_{x_a}^{x_b}\bar{k}\,\psi(x)^2 dx + \int_0^L EI(x)\left[\psi''(x)\right]^2 dx\right.$$

$$\left. - \int_0^L N\left[\psi'(x)\right]^2 dx\right] z\delta z - \left[\int_0^L p_0(x,t)\,\psi(x)\,dx + P_1(t)\,\psi(x_c)\right]\delta z = 0$$

SIMPLIFYING THE FORM

$$\left[m^*\ddot{z} + c^*\dot{z} + k^* z - k_G^* z - p^*(t)\right]\delta z = 0$$

SINCE $\delta z \neq 0$

$$m^*\ddot{z} + c^*\dot{z} + (k^* - k_G^*)\,z = p^*(t)$$

GENERALIZED PARAMETERS

$$m^* = \int_0^L \bar{m}\,\psi(x)^2 dx + M\psi(x_c)^2$$

$$= \bar{m}\int_0^L \left(3\left(\tfrac{x}{L}\right)^2 - \left(\tfrac{x}{L}\right)^3\right)^2 dx + M\left[3\left(\tfrac{x_c}{L}\right)^2 - \left(\tfrac{x_c}{L}\right)^3\right]^2$$

$$= \bar{m}\int_0^L \left[9\frac{x^4}{L^4} - 2(3)\left(\frac{x^2}{L^2}\right)\left(\frac{x^3}{L^3}\right) + \frac{x^6}{L^6}\right] dx + M\left[3(1) - 1\right]^2$$

$$= \bar{m}\int_0^L \left(\frac{9}{L^4}x^4 - \frac{6}{L^5}x^5 + \frac{x^6}{L^6}\right) dx + M(4)$$

$$= \bar{m}\left[\frac{9}{5L^4}x^5 - \frac{x^6}{L^6} + \frac{x^7}{7L^6}\right]_0^L + 4M$$

$$= \bar{m}\left(\frac{9L^5}{5L^4} - 1 + \frac{L^7}{7L^6}\right) + 4M$$

$$= \bar{m}\left(\frac{9}{5}L - 1 + \frac{L}{7}\right) + 4M$$

$$= 1.943\,\bar{m}L - \bar{m} + 4M$$

2.17 Cont.

$$c^* = c\,\psi(x_c)^2$$
$$= c\left[3\left(\frac{x_c}{L}\right)^2 - \left(\frac{x_c}{L}\right)^3\right]^2$$
$$= c\left[3\left(\frac{L}{L}\right)^2 - \left(\frac{L}{L}\right)^3\right]^2$$
$$= 4c$$

$$k^* = \int_{x_a}^{x_b} \bar{k}\,\psi(x)^2\,dx + \int_0^L EI(x)\left[\psi''(x)\right]^2 dx$$
$$= \bar{k}\int_{L/3}^{2L/3}\left(\frac{9}{L^4}x^4 - \frac{6}{L^5}x^5 + \frac{x^6}{L^6}\right)dx + EI\int_0^L\left[\frac{6}{L^2} - \frac{6x}{L^3}\right]^2 dx$$
$$= \bar{k}\left[\frac{9x^5}{5L^4} - \frac{x^6}{L^6} + \frac{x^7}{7L^6}\right]_{L/3}^{2L/3} + EI\int_0^L\left(\frac{36}{L^4} - \frac{72x}{L^5} + \frac{36x^2}{L^6}\right)dx$$
$$= \bar{k}\left[\left[\frac{9}{5L^4}\left(\frac{2L}{3}\right)^5 - \frac{1}{L^6}\left(\frac{2L}{3}\right)^6 + \frac{1}{7L^6}\left(\frac{2L}{3}\right)^7\right] - \left[\frac{9}{5L^4}\left(\frac{L}{3}\right)^5 - \frac{1}{L^6}\left(\frac{L}{3}\right)^6 + \frac{1}{7L^6}\left(\frac{L}{3}\right)^7\right]\right]$$
$$+ EI\left[\frac{36}{L^4}x - \frac{36x^2}{L^5} + \frac{12x^3}{L^6}\right]_0^L$$
$$= \bar{k}\left[(0.2370L - 0.08779 + 0.008361L) - (0.007407L - 0.001372 + 6.532\times10^{-5}L)\right]$$
$$+ EI\left(\frac{36}{L^3} - \frac{36}{L^3} + \frac{12}{L^3}\right)$$
$$= 12\frac{EI}{L^3} + (0.2379L - 0.08642)\bar{k}$$

$$k_G^* = N\int_0^L\left[\psi'(x)\right]^2 dx = 0 \qquad \text{SINCE} \quad N = 0$$

(NEGLECTING SELF-WEIGHT)

$$p^*(t) = \int_0^L p_0(x,t)\,\psi(x)\,dx + P_1(t)\,\psi(x_c)$$
$$= P_0\int_0^L\left(\frac{3x^2}{L^2} - \frac{x^3}{L^3}\right)dx + P_1(t)\left[3\left(\frac{x_c}{L}\right)^2 - \left(\frac{x_c}{L}\right)^3\right]$$
$$= P_0\left[\frac{x^3}{L^2} - \frac{x^4}{4L^3}\right]_0^L + P_1(t)\left[3 - 1\right]$$
$$= P_0\left[L - \frac{L}{4}\right] + 2P_1(t)$$
$$= 0.75\,P_0(t)\,L + 2P_1(t)$$

EQUATION OF MOTION

$$\left[1.943\bar{m}L - \bar{m} + 4M\right]\ddot{z} + \left[4c\right]\dot{z} + \left[12\frac{EI}{L^3} + 0.2379L\bar{k} - 0.08642\bar{k}\right]z$$
$$= 0.75\,P_0(t)\,L + 2P_1(t)$$

NATURAL FREQUENCY

$$\omega = \sqrt{\frac{k^* - k_G^*}{m^*}} = \sqrt{\frac{12\frac{EI}{L^3} + 0.2379\bar{k}L - 0.08642\bar{k}}{1.943\bar{m}L - \bar{m} + 4m}} \quad \text{RAD/SEC}$$

2.17 Cont.

DOWNWARD LOAD N APPLIED:

$$K_G^* = N \int_0^L [\psi'(x)]^2\,dx$$

$$= N \int_0^L \left(\frac{36}{L^4}x^2 - \frac{36}{L^5}x^3 + \frac{9}{L^6}x^4\right)dx$$

$$= N \cdot \left[\frac{12}{L} - \frac{9}{L} + \frac{9}{5L}\right]$$

$$= 4.8\,\frac{N}{L}$$

CALCULATE COMBINED STIFFNESS:

$$K^* - K_G^* = 12\,\frac{EI}{L^3} + 0.2379\,L\bar{k} - 0.08642\,\bar{k} - 4.8\,\frac{N}{L}$$

CALCULATE N_{CR}

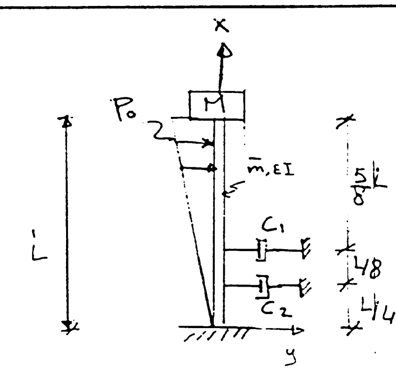
$$K^* - K_{CR}^* = 0$$

$$12\,\frac{EI}{L^3} + 0.2379\,L\bar{k} - 0.08642\,\bar{k} - 4.8\,\frac{N_{CR}}{L} = 0$$

$$N_{CR} = 2.5\,\frac{EI}{L^2} + 0.04956\,\bar{k}L^2 - 1.80\times10^{-2}\,\bar{k}L$$

2.18

$$\phi(x) = \frac{x^2}{L^2}$$



$$M^* = \int_0^L \bar{m}\,\phi^2(x)\,dx + M[\phi(x)]^2\Big|_{x=L}$$

$$= \bar{m}\int_0^L \frac{x^4}{L^4}\,dx + M$$

$$= \bar{m}\,\frac{x^5}{5L^4}\Big|_0^L + M$$

$$\boxed{M^* = \frac{\bar{m}L}{5} + M}$$

2.18 Cont.

$$C^* = C_1 \left[\phi(x)\right]^2_{x=\frac{3}{8}L} + C_2 \left[\phi(x)\right]^2_{x=\frac{L}{4}}$$

$$= C_1 \left.\frac{x^4}{L^4}\right|_{x=\frac{3}{8}L} + C_2 \left.\frac{x^4}{L^4}\right|_{x=\frac{L}{4}}$$

$$\boxed{C^* = \frac{81}{4096} C_1 + \frac{1}{256} C_2}$$

$$K^* = \int_0^L EI \left[\phi''(x)\right]^2 dx$$

$$\phi(x) = \frac{x^2}{L^2} \qquad \phi'' = \frac{2}{L^2} \qquad \left[\phi''\right]^2 = \frac{4}{L^4}$$

$$K^* = EI \int_0^L \frac{4}{L^4}\, dx = 4\frac{EI}{L^4} \left. x \right|_0^L = \frac{4EI}{L^3}$$

<u>note</u> compare with cantilever $K^* = \frac{3EI}{L^3}$ this is because $\phi(x)$ is not accurate enough to descripe the deformed shape (i.e ϕ'' is constant for the whole beam)

$$F^*(t) = \int_0^L \frac{P_0 x}{L} f(t) \frac{x^2}{L^2}\, dx$$

$$= \frac{P_0}{L^3} f(t) \int_0^L x^3\, dx = \frac{P_0 f(t)}{L^3} \left|\frac{x^4}{4}\right|_0^L$$

$$= \frac{P_0 L}{4} f(t)$$

$$K^*_g = N \int_0^L \left(\frac{2x}{L^2}\right)^2 dx$$

$$= N \int_0^L \frac{4x^2}{L^4}\, dx = \frac{4N}{L^4} \left|\frac{x^3}{3}\right|_0^L$$

$$K_g^* = \frac{4}{3}\frac{N}{L}$$

$$\overline{K}^* = K^* - K_g^* \qquad \frac{4EI}{L^3} - \frac{4}{3}\frac{N}{L}$$

2.18 Cont.

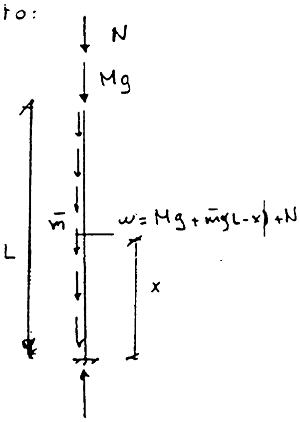
for buckling load $\bar{K}^* = 0$

$$\frac{4EI}{L^3} = \frac{4}{3}\frac{N_{cr}}{L}$$

$$\boxed{N_{cr} = \frac{3EI}{L^2}}$$

Note 1 for cantilever beam $N_{cr} = \frac{\pi^2 EI}{4L^2} = 2.46\frac{EI}{L^2}$

Note 2 if we consider the o.w of the beam & M at the top the K_G is equals to:



$$K_G = \int_0^L [Mg + \bar{m}g(L-x) + N]\frac{4x^2}{L^4}dx$$

$$= \frac{4}{3}\left(\frac{N}{L} + \frac{Mg}{L} + \bar{m}g\right) - \int_0^L \bar{m}g\frac{4x^3}{L^4}dx$$

$$= \frac{4}{3}\left(\frac{N}{L} + \frac{Mg}{L} + \bar{m}g\right) - \bar{m}g$$

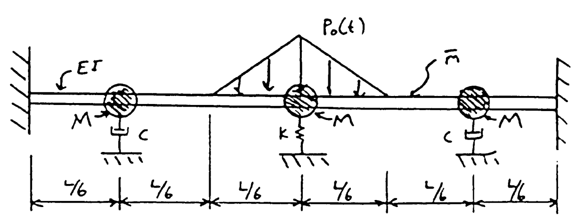
$$= \frac{4}{3}\frac{N}{L} + \frac{4}{3}\frac{Mg}{L} + \frac{\bar{m}g}{3}$$

$$\frac{4}{3}\frac{N}{L} + \frac{4}{3}\frac{Mg}{L} + \frac{\bar{m}g}{3} = \frac{4EI}{L^3}$$

$$\frac{4}{3}\frac{N}{L} = \frac{4EI}{L^3} - \frac{4}{3}\frac{Mg}{L} - \frac{\bar{m}g}{3}$$

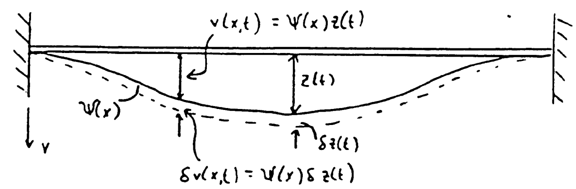
$$\boxed{N_{cr} = \frac{3EI}{L^2} - Mg - \frac{\bar{m}gL}{4}}$$

2.19



2.19 Cont.

SOLUTION



CHECK BOUNDARY CONDITIONS

$$v(0,t) = v(L,t) = v'(0,t) = v'(\tfrac{L}{2},t) = v'(L,t) = 0$$

$$\psi(x) = \frac{16x^2}{L^2} - \frac{32x^3}{L^3} + \frac{16x^4}{L^4} \qquad \psi(0) = 0 \quad \psi(L) = 16 - 32 + 16 = 0$$

$$\psi'(x) = \frac{32x}{L^2} - \frac{96x^2}{L^3} + \frac{64x^3}{L^4} \qquad \psi'(0) = 0 \quad \psi'(L) = \frac{32}{L} - \frac{96}{L} + \frac{64}{L} = 0$$

$$\psi'(\tfrac{L}{2}) = \frac{16}{L} - \frac{24}{L} + \frac{8}{L} = 0$$

EVALUATE THE GENERALIZED PARAMETERS

MASS
$$m^* = \bar{m}\int_0^L \psi(x)^2 dx + M\psi^2(\tfrac{L}{6}) + M\psi^2(\tfrac{L}{2}) + M\psi^2(\tfrac{5L}{6})$$

$$= \bar{m}\int_0^L \left[\frac{256x^4}{L^4} - \frac{1024x^5}{L^5} + \frac{1536x^6}{L^6} - \frac{1024x^7}{L^7} + \frac{256x^8}{L^8}\right]dx$$

$$+ 256M\left[\frac{(\frac{L}{6})^4}{L^4} - \frac{4(\frac{L}{6})^5}{L^5} + \frac{6(\frac{L}{6})^6}{L^6} - \frac{4(\frac{L}{6})^7}{L^7} + \frac{(\frac{L}{6})^8}{L^8}\right]$$

$$+ 256M\left[\frac{(\frac{L}{2})^4}{L^4} - \frac{4(\frac{L}{2})^5}{L^5} + \frac{6(\frac{L}{2})^6}{L^6} - \frac{4(\frac{L}{2})^7}{L^7} + \frac{(\frac{L}{2})^8}{L^8}\right]$$

$$+ 256M\left[\frac{(\frac{5L}{6})^4}{L^4} - \frac{4(\frac{5L}{6})^5}{L^5} + \frac{6(\frac{5L}{6})^6}{L^6} - \frac{4(\frac{5L}{6})^7}{L^7} + \frac{(\frac{5L}{6})^8}{L^8}\right]$$

$$m^* = 0.40635\bar{m}L + 1.1905M$$

DAMPING
$$c^* = c\psi^2(\tfrac{L}{6}) + c\psi^2(\tfrac{5L}{6})$$

$$= 256c\left[\frac{(\frac{L}{6})^4}{L^4} - \frac{4(\frac{L}{6})^5}{L^5} + \frac{6(\frac{L}{6})^6}{L^6} - \frac{4(\frac{L}{6})^7}{L^7} + \frac{(\frac{L}{6})^8}{L^8}\right]$$

$$+ 256c\left[\frac{(\frac{5L}{6})^4}{L^4} - \frac{4(\frac{5L}{6})^5}{L^5} + \frac{6(\frac{5L}{6})^6}{L^6} - \frac{4(\frac{5L}{6})^7}{L^7} + \frac{(\frac{5L}{6})^8}{L^8}\right]$$

$$= 0.1905c$$

STIFFNESS
$$k^* = \int_0^L EI(x)[\psi''(x)]^2 dx + k\psi^2(\tfrac{L}{2})$$

$$= \int_0^L EI(x)\left[\frac{32}{L^2} - \frac{192x}{L^3} + \frac{192x^2}{L^4}\right]^2 dx + 256k\left[\frac{(\frac{L}{2})^4}{L^4} - \frac{4(\frac{L}{2})^5}{L^5} + \frac{6(\frac{L}{2})^6}{L^6} - \frac{4(\frac{L}{2})^7}{L^7} + \frac{(\frac{L}{2})^8}{L^8}\right]$$

$$= EI\int_0^L\left(\frac{1024}{L^4} - \frac{12288x}{L^5} + \frac{49152x^2}{L^6} - \frac{73728x^3}{L^7} + \frac{36864x^4}{L^8}\right)dx + k(1)$$

$$= EI\left[\frac{1024}{L^3} - \frac{12288}{2L^3} + \frac{49152}{3L^3} - \frac{73728}{4L^3} + \frac{36864}{5L^3}\right] + k$$

$$= 204.8\frac{EI}{L^3} + k$$

2.19 Cont.

FORCE $$p^*(t) = \int_{L/3}^{L/2} \left(6p_o \frac{x}{L} - 2p_o\right)\psi(x)\,dx + \int_{L/2}^{2L/3}\left(-6p_o\frac{x}{L} + 4p_o\right)\psi(x)\,dx$$

$$= 2p_o\left[\int_{L/3}^{L/2}\left(3\frac{x}{L} - 1\right)\left(\frac{16x^2}{L^2} - \frac{32x^3}{L^3} + \frac{16x^4}{L^4}\right)dx + \int_{L/2}^{2L/3}\left(-3\frac{x}{L} + 2\right)\left(\frac{16x^2}{L^2} - \frac{32x^3}{L^3} + \frac{16x^4}{L^4}\right)dx\right.$$

$$= 2p_o\left[\int_{L/3}^{L/2}\left(-\frac{16x^2}{L^2} + \frac{80x^3}{L^3} - \frac{112x^4}{L^4} + \frac{48x^5}{L^5}\right)dx + \int_{L/2}^{2L/3}\left(32\frac{x^2}{L^2} - 112\frac{x^3}{L^3} + \frac{128x^4}{L^4} - \frac{48x^5}{L^5}\right)dx\right.$$

$$= 2p_o\left(\left[-\frac{16x^3}{3L^2} + 20\frac{x^4}{L^3} - \frac{112x^5}{5L^4} + 8\frac{x^6}{L^5}\right]_{L/3}^{L/2} + \left[\frac{32x^3}{3L^2} - 28\frac{x^4}{4L^3} + \frac{128x^5}{5L^4} - \frac{8x^6}{L^5}\right]_{L/2}^{2L/3}\right.$$

$$= 2p_o\left(-\frac{16(\frac{L}{2})^3}{3L^2} + \frac{20(\frac{L}{2})^4}{L^3} - \frac{112(\frac{L}{2})^5}{5L^4} + \frac{8(\frac{L}{2})^6}{L^5} + \frac{16(\frac{L}{3})^3}{3L^2} - \frac{20(\frac{L}{3})^4}{L^3} + \frac{112(\frac{L}{3})^5}{5L^4} - \frac{8(\frac{L}{3})^6}{L^5}\right.$$

$$\left. + \frac{32(\frac{2L}{3})^3}{3L^2} - \frac{28(\frac{2L}{3})^4}{4L^3} + \frac{128(\frac{2L}{3})^5}{5L^4} - \frac{8(\frac{2L}{3})^6}{L^5} - \frac{32(\frac{L}{2})^3}{3L^2} + \frac{28(\frac{L}{2})^3}{4L^3} - \frac{128(\frac{L}{2})^5}{5L^4} + \frac{8(\frac{L}{2})^6}{L^5}\right)$$

$$= 6.7069\, p_o(t)\, L$$

EQUATION OF MOTION

$$(0.40635\bar{m}L + 1.1905M)\ddot{z} + (0.1905c)\dot{z} + \left(204.8\frac{EI}{L^3} + k\right)z = 6.7069p_o(t)L$$

NATURAL FREQUENCY

$$\omega = \sqrt{\frac{204.8\frac{EI}{L^3} + k}{0.40635\bar{m}L + 1.1905M}}$$

GENERALIZED GEOMETRIC STIFFNESS

$$K_G^* = N\int_0^L [\psi'(x)]^2\,dx$$

$$= N\int_0^L\left(32\frac{x}{L^2} - 96\frac{x^2}{L^3} + 64\frac{x^3}{L^4}\right)^2 dx$$

$$= 1024N\int_0^L\left(\frac{x^2}{L^4} - \frac{6x^3}{L^5} + \frac{13x^4}{L^6} - \frac{12x^5}{L^7} + \frac{4x^6}{L^8}\right)dx$$

$$= 1024N\left[\frac{1}{3L} - \frac{6}{4L} + \frac{13}{5L} - \frac{2}{L} + \frac{4}{7L}\right]$$

$$= 4.8762\frac{N}{L}$$

COMBINED STIFFNESS

$$k^* - k_G^* = 204.8\frac{EI}{L^3} + k - 4.8762\frac{N}{L}$$

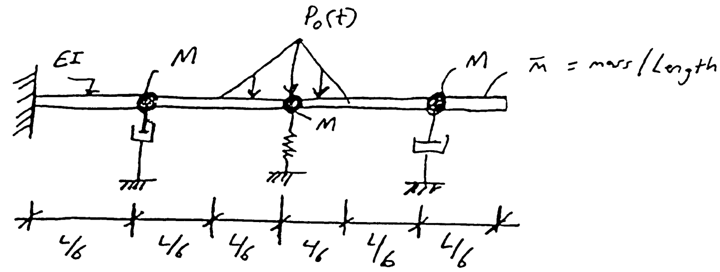
CRITICAL BUCKLING LOAD, N_{CR}

$$K^* - K_G^* = 0$$

$$204.8\frac{EI}{L^3} + k - 4.8762\frac{N_{CR}}{L} = 0$$

$$N_{CR} = 42\frac{EI}{L^2} + 0.20508\,kL$$

2.20



$\psi(x) = 3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3$

Use deflection at the free end as the generalized coordinate

SOLUTION:

$$\psi(x) = \frac{3x^2}{L^2} - \frac{x^3}{L^3} \qquad [\psi(x)]^2 = \frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}$$

$$\psi'(x) = \frac{6x}{L^2} - \frac{3x^2}{L^3} \qquad [\psi'(x)]^2 = \frac{36x^2}{L^4} - \frac{36x^3}{L^5} + \frac{9x^4}{L^6}$$

$$\psi''(x) = \frac{6}{L^2} - \frac{6x}{L^3} \qquad [\psi''(x)]^2 = \frac{36}{L^4} - \frac{72x}{L^5} + \frac{36x^2}{L^6}$$

$$m^* = \int_0^L m(x)\,\psi^2(x)\,dx + \sum M_i \psi^2(x_i)$$

$$m^* = \bar{m}\int_0^L \left(\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right)dx + M\left[\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right]_{L/6,\ L/2,\ 5L/6}$$

$$m^* = \bar{m}\left[\frac{9x^5}{5L^4} - \frac{x^6}{L^5} + \frac{x^7}{7L^6}\right]_0^L + M\left[\frac{9(\frac{L}{6})^4}{L^4} - \frac{6(\frac{L}{6})^5}{L^5} + \frac{(\frac{L}{6})^6}{L^6}\right]$$

$$+ M\left[\frac{9(\frac{L}{2})^4}{L^4} - \frac{6(\frac{L}{2})^5}{L^5} + \frac{(\frac{L}{2})^6}{L^6}\right]$$

$$+ M\left[\frac{9(\frac{5L}{6})^4}{L^4} - \frac{6(\frac{5L}{6})^5}{L^5} + \frac{(\frac{5L}{6})^6}{L^6}\right]$$

$$\boxed{m^* = .943\,\bar{m}L + 2.66\,M}$$

$$c^* = \sum c\psi^2(x) = c\left[\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right]_{L/6,\ 5L/6}$$

$$= c\left[\frac{9(\frac{L}{6})^4}{L^4} - \frac{6(\frac{L}{6})^5}{L^5} + \frac{(\frac{L}{6})^6}{L^6}\right]$$

$$+ c\left[\frac{9(\frac{5L}{6})^4}{L^4} - \frac{6(\frac{5L}{6})^5}{L^5} + \frac{(\frac{5L}{6})^6}{L^6}\right]$$

$$\boxed{c^* = 2.27c}$$

2.20 Cont.

$$k^* = \int EI(x)\left[\psi''(x)\right]^2 dx + \sum k_i \psi^2(x_i)$$

$$= EI\int_0^L \left(\frac{36}{L^4} - \frac{72x}{L^5} + \frac{36x^2}{L^6}\right) dx + k\left[\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right]^{L/2}$$

$$= EI\left[\frac{36x}{L^4} - \frac{72x^2}{2L^5} + \frac{36x^3}{3L^6}\right]_0^L + k\left[\frac{9\left(\frac{L}{2}\right)^4}{L^4} - \frac{6\left(\frac{L}{2}\right)^5}{L^5} + \frac{\left(\frac{L}{2}\right)^6}{L^6}\right]$$

$$= EI\left[\frac{36}{L^3} - \frac{36}{L^3} + \frac{12}{L^3}\right] + k(.391)$$

$$\boxed{k^* = \frac{12EI}{L^3} + .391K}$$

$$p^*(t) = \int p(x,t)\psi(x)\,dx + \sum_i p_i \psi(x_i) \quad 0$$

$$= \int_{L/3}^{2L/3} p_0(t)\left[\frac{3x^2}{L^2} - \frac{x^3}{L^3}\right] dx$$

$$= p_0\left[\frac{x^3}{L^2} - \frac{x^4}{4L^3}\right]_{L/3}^{2L/3}$$

$$= p_0\left[\left(\frac{\left(\frac{2L}{3}\right)^3}{L^2} - \frac{\left(\frac{2L}{3}\right)^4}{4L^3}\right) - \left(\frac{\left(\frac{L}{3}\right)^3}{L^2} - \frac{\left(\frac{L}{3}\right)^4}{4L^3}\right)\right]$$

$$\boxed{p^*(t) = .213\, p_0 L}$$

Eq. of Motion

$$(.943\,\bar{m}L + 2.66M)\ddot{z} + (2.27c)\dot{z} + \left(\frac{12EI}{L^3} + .391K\right)z$$

$$= .213\, p_0 L$$

Natural Frequency

$$\omega = \sqrt{\frac{k^*}{m^*}} = \sqrt{\frac{\left(\frac{12EI}{L^3}\right) + .391K}{.943\,\bar{m}L + 2.66M}}$$

$$k_G^* = N\int \left(\psi'(x)\right)^2 dx$$

$$= N\int_0^L \left[\frac{36x^2}{L^4} - \frac{36x^3}{L^5} + \frac{9x^4}{L^6}\right]_0^L$$

$$= N\left[\frac{12x^3}{L^4} - \frac{9x^4}{L^5} + \frac{9x^5}{5L^6}\right]_0^L$$

$$= N\left[\frac{12}{L} - \frac{9}{L} + \frac{1.8}{L}\right] \Rightarrow k_G^* = 4.8\frac{N}{L}$$

2.20 Cont.

Combined Stiffness

$$K^{*} - KG^{*} = \frac{12EI}{L^3} + .391K - \frac{4.8 N_{cr}}{L} = 0$$

$$\boxed{N_{cr} = 2.5\frac{EI}{L^2} + .08145\,KL}$$