CHAPTER 1

Introduction to Ordinary Differential Equations Solutions

We have made every effort possible to ensure that these solutions are complete, clear, and error-free. However, since the book has over 2800 problems, we may have made a mistake somewhere. If you find any errors, omissions, or solutions that just don't seem clear enough please let us know and we will try to get a correction up right away.

1.1 In this problem you are going to confirm that the solution to the differential equation $dy/dx = y\cos x$ with condition $y(0) = 1$ is $y = e^{\sin x}$.

(a) If $y = e^{\sin x}$, what is dy/dx?

Final Answer: $e^{\sin} x \times \cos x$

(b) If $y = e^{\sin x}$, what is $y\cos x$?

Final Answer: $e^{\sin} x \times \cos x$

If your answers to Parts a and b are not the same, that would indicate that this function does *not* solve this differential equation. In this case, we are assuring you that it does—so if they did not come out the same, go back and find your mistake.

(c) Now plug $x = 0$ into the function to make sure it gives the answer specified in the initial condition.

Solution: $e^{\sin 0} = e^0 = 1$

1.2 One valid solution to the differential equation $f''(x) + 2f'(x) = 2e^x$ is $f(x) = (2/3)(e^{-2x} + e^x)$.

(a) Demonstrate that $f(x)$ does, in fact, solve the differential equation.

Solution:

$$f'(x) = \frac{2}{3}(-2e^{-2x} + e^x)$$
$$f''(x) = \frac{2}{3}(4e^{-2x} + e^x)$$
$$f''(x) + 2f'(x) = \frac{8}{3}e^{-2x} + \frac{2}{3}e^x - \frac{8}{3}e^{-2x} + \frac{4}{3}e^x = 2e^x$$

(b) If you add three to $f(x)$, is the resulting function also a solution?

Solution:

$$f(x) = \frac{2}{3}(e^{-2x} + e^x) + 3$$
$$f'(x) = \frac{2}{3}(-2e^{-2x} + e^x)$$
$$f''(x) = \frac{2}{3}(4e^{-2x} + e^x)$$
$$f''(x) + 2f'(x) = \frac{8}{3}e^{-2x} + \frac{2}{3}e^x - \frac{8}{3}e^{-2x} + \frac{4}{3}e^x = 2e^x$$

Since the differential equation involves f' and f'' but not f, and the derivatives are unchanged by adding three to f, this is still a solution.

Final Answer: Yes

(c) If you multiply $f(x)$ by 2, is the resulting function also a solution?

Solution:

$$f(x) = \frac{4}{3}(e^{-2x} + e^x)$$
$$f'(x) = \frac{4}{3}(-2e^{-2x} + e^x)$$
$$f''(x) = \frac{4}{3}(4e^{-2x} + e^x)$$
$$f''(x) + 2f'(x) = \frac{16}{3}e^{-2x} + \frac{4}{3}e^x - \frac{16}{3}e^{-2x} + \frac{8}{3}e^x = 4e^x$$

Every term on the left gets multiplied by 2 when you multiply f by 2, but the right side doesn't change, so the two are no longer equal.

Final Answer: No

(d) You now have two valid solutions of this differential equation. One of those functions also satisfies the initial condition $f(0) = 13/3$. Which one?

Solution: The two valid solutions are $f_1(x) = (2/3)(e^{-2x} + e^x)$ and $f_2(x) = (2/3)(e^{-2x} + e^x) + 3$. Plugging in $x = 0$ gives $f_1(0) = (4/3)$ and $f_2(0) = (13/3)$, so the solution with this initial condition is $(2/3)(e^{-2x} + e^x) + 3$.

Final Answer: $(2/3)(e^{-2x} + e^x) + 3$

1.3 $dy/dx = 3$

(a) $y = 3$

(b) $y = 3x$

(c) $y = 3x + 9$

(d) $y = (3/2)x^2$

(e) $y = e^{3x}$

Final Answer: (b), (c)

1.4 $dy/dx = 3x$

(a) $y = 3$

(b) $y = 3x$

(c) $y = 3x + 9$

(d) $y = (3/2)x^2$

(e) $y = e^{3x}$

Final Answer: (d)

1.5 $dy/dx = 3y$

(a) $y = (3/2)x^2$

(b) $y = e^{3x}$

(c) $y = 3e^x$

(d) $y = 7e^{3x}$

(e) $y = e^{3x+7}$

(f) $y = e^{3x} + 7$

Final Answer: (b), (d), (e)

1.6 $dz/dt = z^2$

(a) $z = (1/3)t^3$

(b) $z = e^{t^2}$

(c) $z = 1/t$

(d) $z = -1/t$

(e) $z = -1/t + 6$

(f) $z = -1/(t + 6)$

Final Answer: (d), (f)

1.7 $dP/dr = P - r$

(a) $P = r + 1$

(b) $P = e^r$

(c) $P = e^r + r + 1$

(d) $P = e^r + r + 2$

(e) $P = 2e^r + r + 1$

Final Answer: (a), (c), (e)

1.8 $dy/dx = (y^2 - xy)/(x^2 \ln x + 2xy)$

(a) $y = \ln x + x^2$

(b) $y = (\ln x)/x$

(c) $y = -x \ln x$

Final Answer: (c)

1.9 $d^2x/dt^2 = x$

(a) $x = e^t$

(b) $x = e^{-t}$

(c) $x = -e^t$

(d) $x = 3e^t - 5e^{-t}$

(e) $x = 0$

Final Answer: All of them work

1.10 $d^2x/dt^2 = -x$

(a) $x = e^t$

(b) $x = e^{-t}$

(c) $x = -e^t$

(d) $x = \sin t$

(e) $x = \cos t$

(f) $x = 10\sin t + 15\cos t$

Final Answer: (d), (e), (f)

1.11 $d^2y/dx^2 - 6(dy/dx) + 9y = 0$

(a) $y = e^{3x}$

(b) $y = e^{-3x}$

(c) $y = 3e^{x}$

(d) $y = 2xe^{3x}$

(e) $y = 7e^{3x} - 5xe^{3x}$

Final Answer: (a), (d), (e)

1.12 $d^2y/dx^2 - 6(dy/dx) + 9y = 18$

(a) $y = 2$
(b) $y = e^{3x}$
(c) $y = e^{3x} + xe^{3x} + 2$

Final Answer: (a), (c)

1.13 $dh/dz = \ln z$, $h(1) = 5$

(a) $h = 1/z + 4$

(b) $h = \ln z + 5$

(c) $h = z \ln z - z + 5$

(d) $h = z \ln z - z + 6$

Final Answer: (d)

1.14 $dy/dx = y/x, y(3) = 12$

(a) $y = x$

(b) $y = 4x$

(c) $y = x + 9$

(d) $y = 12$

Final Answer: (b)

1.15 $dx/dt = t\sqrt{x}, x(0) = 9$

(a) $x = t^2\sqrt{t} + 9$

(b) $x = \sqrt{t^2 - 3\sqrt{t}} + 9$

(c) $x = (t^2/4 + 3)^2$

(d) $x = 9/(t^2 + 1)$

Final Answer: (c)

1.16 $dy/dt = 2y(1-y), y(0) = 1/10$

(a) $y = e^{2t}/(9 + e^{2t})$

(b) $y = e^{2t}/10$

(c) $y = (1/10)(e^{2t} - 1)$

(d) $y = 2t(1-t) + 10$

Final Answer: (a)

1.17 You are running an online tournament with $P(t)$ players participating. Match each description below with the appropriate differential equation. (Nothing is changing other than what is described. For example, in Part (a) no players are being eliminated, while in Part (b) no new ones are joining.) Each equation should be matched to only one scenario.

(a) Each day 50 new players join.	I. $dP/dt = 50$
(b) Each day half the players in the tournament are eliminated.	II. $dP/dt = -50$
(c) Each day 50 players are eliminated.	III. $dP/dt = 3P$
(d) Each day everyone in the tournament convinces three friends to join.	IV. $dP/dt = 3t$
(e) Three players join the first day, six the second day, nine the third day, and so on.	V. $dP/dt = -P/2$

Final Answer: (a) I. (b) V. (c) II. (d) III. (e) IV.

1.18 Parts (a)–(e) below give five different scenarios for the evolution of a rabbit population $R(t)$. Match each one with the appropriate differential equation. Each equation should be matched to only one scenario.

(a) The rabbit population is constant.	I. $R = 0$
(b) 10 new rabbits are born every year.	II. $dR/dt = 10R$
(c) On average, each rabbit produces 10 new rabbits per year.	III. $dR/dt = 10$
(d) Each year each rabbit produces 10 new rabbits, but 100 rabbits are killed.	IV. $dR/dt = 0$
(e) There are no rabbits.	V. $dR/dt = 10R - 100$

Final Answer: (a) IV. (b) III. (c) II. (d) V. (e) I.

1.19 An object of mass m is moving along the x-axis (one dimension only). In each part of this problem you will be told the force acting on this object, and you will write a differential equation for the position $x(t)$ of the object. You do *not* need to solve the equation.

(a) $F = -mg$ where g is a constant.
Solution: $F = ma = m(d^2x/dt^2)$. Setting this equal to $-mg$ gives you $m(d^2x/dt^2) = -mg$. So the differential equation is $d^2x/dt^2 = -g$.
Final Answer: $d^2x/dt^2 = -g$

(b) A charged ball experiences a force towards the origin proportional to one over its distance from the origin squared.
Solution: For the force to point toward the origin, F must point in the positive direction whenever $x(t)$ is negative and in the negative direction whenever $x(t)$ is positive. Therefore, the sign of the force must be opposite that of the position $x(t)$ of the charged ball. If you let k be the positive constant of proportionality, the equation for the force is $F = -k|x|/x^3$. Since $F = m(d^2x/dt^2)$, putting the two equations together gives you $m(d^2x/dt^2) = -k|x|/x^3$. (You can avoid using the absolute value by writing the function piecewise: $m(d^2x/dt^2) = -k/x^2$ for $x > 0$ and $m(d^2x/dt^2) = k/x^2$ for $x < 0$)
Final Answer: $m(d^2x/dt^2) = -k|x|/x^3$

(c) A boat experiences a constant forward force due to its motor and a backward drag force proportional to its speed.
Solution: $F = k - bv$, where k and b are positive constants, so $m(d^2x/dt^2) = k - b(dx/dt)$.
Final Answer: $m(d^2x/dt^2) = k - b(dx/dt)$

(d) A cyclist experiences a constant backward force due to friction and a forward force from pedaling. As the cyclist gets tired the pedaling force decreases linearly with time. (If the cyclist stops moving the frictional force will stop too, but ignore that here.)
Solution: $F = (b - rt) - k$, where b, r, and k are positive constants, so $m(d^2x/dt^2) = (b - rt) - k$.
Final Answer: $m(d^2x/dt^2) = (b - rt) - k$

1.20 The function $f(x)$ has the peculiar property that its *derivative* is the same function as its *square root*. (Assume $f(x) \geq 0$ for all x.)

(a) Express this sentence—"The derivative of $f(x)$ and the square root of $f(x)$ are the same function"—as a differential equation.
Final Answer: $df/dx = \sqrt{f}$

(b) Show that $f(x) = 0$ is a solution but $f(x) = 1$ is not.
Solution: If $f(x) = 0$, then both $f'(x) = 0$ and $\sqrt{f(x)} = \sqrt{0} = 0$. If $f(x) = 1$, then $f'(x) = 0$ but $\sqrt{f(x)} = \sqrt{1} = 1$. Since $f'(x)$ does not equal $\sqrt{f(x)}$ for $f(x) = 1$, it is not a solution.

(c) Now assume that $f(0) = 1$. Is $f(x)$ an increasing function, a decreasing function, or flat? How can you tell?
Solution: At the point $(0, 1)$, the slope is $f'(0) = \sqrt{f} = 1$, so the function is increasing with a slope of 1.
Final Answer: increasing

(d) Is $f(x)$ concave up, concave down, or neither? How can you tell?
Solution: As $f(x)$ gets higher, $\sqrt{f(x)}$ gets higher, and so $f'(x)$ gets higher. This means the slope keeps increasing, so $f(x)$ is concave up.
Final Answer: concave up

1.21 The Explanation (Section 2.2) discussed two financial situations: one in which Shannon earns money at a consistent (linear) rate, and one in which she continuously earns interest on her money. Now suppose both are going on at the same time. Shannon makes \$30,000/year salary, *and* her money always goes immediately into the bank, where it earns interest at an annual rate of 10%.

(a) Write the differential equation that says "Shannon's money increases every year by \$30,000 plus 10% of whatever she has." Use M for the total of her money and t for time in years. (To make the answer look nicer, measure M in "thousands of dollars" instead of "dollars.")
Final Answer: $dM/dt = 30 + M/10$

(b) Assuming Shannon starts with no money, show that the function $M(t) = 300(e^{t/10} - 1)$ satisfies both the differential equation and the initial condition.
Solution:

$$\frac{dM}{dt} = 300\left(\frac{1}{10}e^{t/10}\right) = 30e^{t/10}$$
$$30 + \frac{M}{10} = 30 + 30(e^{t/10} - 1) = 30 + 30e^{t/10} - 30$$

They come out the same, so the differential equation is satisfied.
Final Answer: $M(0) = 300(e^0 - 1) = 0$ so the initial condition is satisfied.

(c) We were able to write our solution in that form because we had already specified what units we were using for time and money. To write this solution with correct units, so that it would be valid for any units of time and money, we can write $M(t) = D(e^{t/k} - 1)$, where $D = 300$ and $k = 10$. What are the units of D and k?
Final Answer: D is in dollars, k in years

1.22 Little Benjamin sneaks onto his father's computer every night. His father thinks Ben is playing video games. Actually, Ben is day trading, and he is very good at it. However much money he has at the beginning of the week, he always earns 10% of that in a week's time. Unfortunately, Ben also has a bubble-gum habit that costs him \$15 a week.

(a) Write a differential equation that represents Ben's net worth. Use M for his money and t for time (measured in weeks).
Final Answer: $dM/dt = M/10 - 15$

(b) How much money does Ben need in order to break even every week? ("Break even" means his net worth is neither increasing nor decreasing; in other words, dM/dt is zero.)
Solution: $M/10 - 15 = 0$ so $M = 150$
Final Answer: \$150

(c) If Ben starts off with *more* money than you calculated in Part b, what will happen to his money over time? Explain how your answer is based on your differential equation in Part a.
Solution: If $M > 150$ then $M/10 - 15 > 0$ so dM/dt is positive. That means Ben is making money. The more money he has, the higher dM/dt gets, so his money will increase faster and faster.
Final Answer: make money faster and faster

(d) If Ben starts off with *less* money than you calculated in Part b, what will happen to his money over time? Explain how your answer is based on your differential equation in Part a.
Solution: If $M < 150$ then $M/10 - 15 < 0$ so dM/dt is negative. That means Ben is losing money. The less money he has, the more negative dM/dt gets, so he will rapidly run out of money.
Final Answer: quickly run out of money

1.23 Mary starts her career with an initial salary S_0. Each year she gets a 5% raise, so her salary one year later is $1.05 \times S_0$, her salary the following year is $1.05 \times 1.05 \times S_0 = 1.05^2 \times S_0$, and so on.

(a) Write an expression for Mary's salary $S(t)$. This will *not* be a differential equation, but a simple function of time.
Solution: $S(t) = S_0 1.05^t$

(b) Assuming Mary never gains or loses money in any way other than earning her salary, write a differential equation for her amount of money $M(t)$. (You will need to use your answer to Part a here.)
Solution: $dM/dt = S_0 1.05^t$

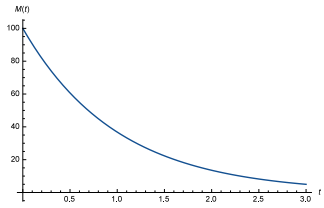
(c) Now assume that in addition to earning her salary, Mary also spends an amount E each year, which starts at E_0 and increases by 2% each year (because of inflation). Write a new differential equation for $M(t)$.
Solution: $dM/dt = S_0 1.05^t - E_0 1.02^t$

(d) Finally, assume that Mary earns a salary $S(t)$, spends an amount $E(t)$, *and* is putting her money in the bank where it is earning 4% interest each year. Write a differential equation for $M(t)$.
Final Answer: $dM/dt = S_0 1.05^t - E_0 1.02^t + .04M$

1.24 The chemical tetrahydrofrefurol decomposes in the presence of a biological agent. Research shows that the rate of decomposition is proportional to the remaining mass. When the biological agent is first introduced and the decomposition begins ($t = 0$), there are 100 kg of tetrahydrofrefurol.

Use M for the mass of tetrahydrofrefurol and t for the amount of time since it began decomposing.

(a) Sketch a possible graph for the function $M(t)$ on purely physical grounds. You don't need to write any equations yet; just make sure your graph represents the idea that the more you have, the more you lose.
Solution: The graph needs to always decrease, but as the amount decreases the rate of decrease slows, so the curve should be concave up and asymptotically approach zero.



(b) Write a differential equation that represents the sentence "The amount of mass that disappears in any given hour is proportional to the mass at that hour." Your equation will introduce a constant of proportionality k.
Final Answer: $dM/dt = kM$ or $dM/dt = -kM$

(c) What are the units of k? Is it positive or negative? How do you know?
Solution: M has units of kg and dM/dt has units of kg/hour, so the units on k must be "per hours" or hr^{-1}.
M is a mass which is always positive, and dM/dt is negative because the mass is decreasing. So if your differential equation is $dM/dt = -kM$ then k is positive; if you write $dM/dt = km$ then k is negative.
Most engineers would write $dM/dt = -kM$ in order to make k come out positive. This does not change the math in any fundamental way, but it does make it easier to interpret the answer.
Final Answer: The units on k are hr^{-1}. If you wrote $dM/dt = km$ then k is negative; if you wrote $dM/dt = -kM$ then k is positive.

(d) The function $M(t) = 0$ should be a valid solution of your differential equation. Describe the scenario described by this solution.
Solution: If you start with no mass, you have no mass forever.

(e) Find a function *other than* $M(t) = 0$ that solves your differential equation.
Final Answer: For $dM/dt = kM$ a solution is $M(t) = e^{kt}$. For $dM/dt = -kM$ a solution is $M(t) = e^{-kt}$.

(f) Find a function that solves your differential equation *and* has the property that $M(0) = 100$. (If the graph of this function doesn't look more or less like the graph you drew in Part a, one of them is wrong; figure out which one, and fix it!)
Final Answer: For $dM/dt = kM$ the solution is $M(t) = 100e^{kt}$. For $dM/dt = -kM$ the solution is $M(t) = 100e^{-kt}$.

(g) A different chemical follows the same differential equation, but with a much higher value of k. Compare its decay process to the decay of tetrahydrofrefurol.
Solution: It will decay faster. We can see that from the differential equation itself: the same M will lead to a more negative dM/dt. We can also see it from the solution: e^{-3t} decays faster than e^{-2t}.

1.25 As a snowball melts, its mass m decreases at a rate proportional to its surface area. The surface area of a snowball is directly proportional to $m^{2/3}$.

(a) The problem tells us that dm/dt is proportional to $m^{2/3}$. Is the constant of proportionality positive or negative? How do you know?
Solution: Because the snowball is melting, dm/dt is negative. The mass and surface area are of course positive by definition. So the constant of proportionality must be negative.
Final Answer: negative

(b) Write the differential equation. If the constant of proportionality is positive, call it k^2. If it is negative, call it $-k^2$.
Final Answer: $dm/dt = -k^2 m^{2/3}$

(c) Show that $m = (5 - k^2 t/3)^3$ is a valid solution to your differential equation.
Solution:

$$\frac{dm}{dt} = 3\left(5 - \frac{k^2 t}{3}\right)^2 \times \left(-\frac{k^2}{3}\right) = -k^2\left(5 - \frac{k^2 t}{3}\right)^2$$

$$-k^2 m^{2/3} = -k^2\left(\left(5 - \frac{k^2 t}{3}\right)^3\right)^{2/3} = -k^2\left(5 - \frac{k^2 t}{3}\right)^2$$

They come out the same, so the differential equation is satisfied.

(d) What initial condition is implied by that solution?
Solution: $m(0) = (5 + 0)^3 = 125$
Final Answer: 125

1.26 A mass on a spring experiences a force $F_{\text{spring}} = -kx$ where x is the displacement from the relaxed position and k is a positive constant. It also experiences the force of friction from the table the mass is sliding along: the force is $F_{\text{friction}} = -3v/|v|$.

(a) $-3v/|v|$? That's a strange looking force. What is its magnitude? Which way does it push?
Final Answer: Its magnitude is a constant 3, and it pushes in the opposite direction of the velocity at any given time.

(b) Using Newton's Second Law $F_{\text{total}} = ma$ and the definitions of velocity (dx/dt) and acceleration (d^2x/dt^2), write a second-order differential equation for the mass's position as a function of time $x(t)$.
Final Answer: $m(d^2x/dt^2) = -kx - 3(dx/dt)/|dx/dt|$

(c) In general, the best way to work with absolute values mathematically is to split into two cases. When $v \geq 0$ you can replace $|v|$ with v in your differential equation. Show that in this case one valid solution to the differential equation is:

$$x = 2\sin\left(\sqrt{\frac{k}{m}}\,t + \frac{\pi}{4}\right) - \frac{3}{k}$$

Solution: The differential equation in this case is $d^2x/dt^2 + (k/m)x = -3/m$. Plugging in this solution gives:
$-2(k/m)\sin\left(\sqrt{k/m}\,t + \pi/4\right) + 2(k/m)\sin\left(\sqrt{k/m}\,t + \pi/4\right) - 3/m = 3/m$ which works.

(d) When $v < 0$ you can replace $|v|$ in your differential equation with $-v$. Find a solution to the differential equation that works in this case. (The technique of choice here is trial-and-error: play around until you find a function that works!)
Final Answer: $2\sin\left(\sqrt{k/m}\,t + \pi/4\right) + 3/k$

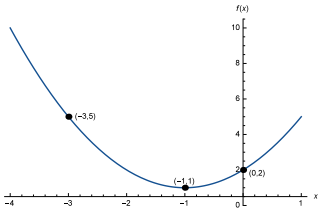
1.27 Let $y = f(x)$ represent one particular graph (not the only one!) that follows the differential equation $dy/dx = y - x^2$.

(a) $f(x)$ goes through the point $(-1, 1)$. What is the slope of the curve at that point?
Final Answer: 0

(b) $f(x)$ also goes through the point $(-3, 5)$. What is the slope of the curve at that point?
Final Answer: -4

(c) $f(x)$ also goes through the point $(0, 2)$. What is the slope of the curve at that point?
Final Answer: 2

(d) Based on all the information you have accumulated, draw a possible sketch of $f(x)$.
Solution:



1.28 Let $y = f(x)$ represent one particular graph (not the only one!) that follows the differential equation $dy/dx = (x + 2y)/(x + 2)$.

(a) $f(x)$ goes through the point $(-3/2, 3/4)$. What is the slope of the curve at that point?

Solution: Plug $x = -3/2$ and $y = 3/4$ into the formula for dy/dx to find the slope at that point, and you get 0.

Final Answer: 0

(b) Find d^2y/dx^2 as a function of x and y. In other words, find the derivative with respect to x of dy/dx. When you first apply the quotient rule, your answer will have dy/dx in it. You can then substitute $dy/dx = (x + 2y)/(x + 2)$ to obtain a function of just x and y. Simplify as much as possible.

Solution:

$$\frac{d^2y}{dx^2} = \frac{(1 + 2(dy/dx))(x + 2) - (x + 2y)(1)}{(x + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1 + 2(\frac{x+2y}{x+2}))(x + 2) - (x + 2y)(1)}{(x + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2(x + y + 1)}{(x + 2)^2}$$

Final Answer: $d^2y/dx^2 = 2(x + y + 1)/(x + 2)^2$

(c) At the point $(-3/2, 3/4)$, is the function concave up or concave down?

Solution:

$$\frac{d^2y}{dx^2} = \frac{2(-3/2 + 3/4 + 1)}{(-3/2 + 2)^2} = 2 > 0$$

So the function is concave up.

Final Answer: Concave up

(d) Based on your answers to a and c, what can you conclude about this function at the point $(-3/2, 3/4)$?

Solution: $f' = 0$ tells you that $(-3/2, 3/4)$ is a critical point, and $f'' > 0$ tells you that this critical point is a local minimum of the function f.

Final Answer: local minimum

1.29 A quantity Q evolves over time according to the differential equation $dQ/dt = Q(Q-3)(Q-5)$.

(a) If $Q(0) = 3$ then, at that point, $dQ/dt = 0$. So Q will stay 3, which means dQ/dt will stay 0 forever. Therefore $Q = 3$ is referred to as an "equilibrium solution" of the differential equation. What are the two other equilibrium solutions?
Final Answer: $Q = 0$ and $Q = 5$

(b) If Q is between 0 and 3, is dQ/dt positive or negative? Based on this, how do you expect Q to evolve over time?
Solution: Q is positive, $(Q-3)$ is negative, and $(Q-5)$ is negative, so dQ/dt must be positive. This tells you that Q increases with time, but starts to level off as the graph approaches $Q = 3$. (Since $dQ/dt = 0$ at $Q = 3$, the slope will approach 0 as the curve gets closer to $Q = 3$.)

(c) If Q is between 3 and 5, is dQ/dt positive or negative? Based on this, how do you expect Q to evolve over time?
Solution: dQ/dt is negative. This tells you that Q decreases with time, but starts to level off as the graph approaches $Q = 3$.

(d) If $Q > 5$, is dQ/dt positive or negative? Based on this, how do you expect Q to evolve over time?
Solution: Since all the terms are positive, dQ/dt is positive. This tells you that Q will increase, and since $(dQ/dt) > 0$ for all values of $Q > 5$, the graph of Q will keep on increasing indefinitely.

1.30 Two different objects are both moving along the x-axis. Both experience forces that are proportional to their distance from $x = 0$, so we can write $F = kx$. Since $F = ma$ and $a = d^2x/dt^2$, we can write $d^2x/dt^2 = (k/m)x$. However, there is one big difference. For Object A, $k/m = 1$; for Object B, $k/m = -1$.

(a) For Object A, $d^2x/dt^2 = x$. Based directly on this differential equation, if the object starts at rest at $x = 0$, what will its acceleration be? What will happen over time?
Solution: $d^2x/dt^2 = 0$. Since the object starts at rest and its velocity is not changing, the object will remain at $x = 0$ forever.

(b) If Object A starts at rest at $x = 1$, will it accelerate to the right or left? Where will it go next, and what will happen to its acceleration? What will happen over time?
Solution: $d^2x/dt^2 = 1$, so it will accelerate to the right. The object will start to move to the right, and the further right the object goes, the greater its acceleration to the right becomes. So the object will keep going to the right at an ever increasing speed. (In the words of Buzz Lightyear: To infinity and beyond!)

(c) If Object A starts at rest at $x = -1$, what will its acceleration be? Where will it go next, and what will happen to its acceleration? What will happen over time?
Solution: $d^2x/dt^2 = -1$, so it will accelerate to the left. The object will start to move to the left, and as it moves to the left, its acceleration will become more negative. The further left the object goes, the more negative its acceleration becomes, so the object will keep rolling off to the left at an ever increasing speed. (To negative infinity and beyond!)

(d) For Object B, $d^2x/dt^2 = -x$. If the object starts at rest at $x = 0$, what will its acceleration be? What will happen over time?
Solution: $d^2x/dt^2 = 0$. Since the object starts at rest and its velocity is not changing, the object will remain at $x = 0$ forever.

(e) If Object B starts at rest at $x = 1$, will it accelerate to the right or left? Where will it go next, and what will happen to its acceleration? What will happen over time?
Solution: $d^2x/dt^2 = -1$, so it will accelerate to the left. As the object begins to move left towards the origin, its acceleration will become less negative, but its speed to the left will still keep increasing. When the object goes past $x = 0$ into the negative x-values, the acceleration will become positive and start pointing to the right. This will slow the object down until it comes to a full stop and starts moving back toward the right. When the object crosses the $x = 0$ again, the acceleration will switch sign again and start pulling the object back toward the left. The object will therefore oscillate back and forth indefinitely. (You may recognize the differential equation in this case as that of a simple harmonic oscillator.)

(f) If Object B starts at rest at $x = -1$, what will its acceleration be? Where will it go next, and what will happen to its acceleration? What will happen over time?
Solution: $d^2x/dt^2 = 1$, so it will accelerate to the right, toward the origin. From here, the behavior is the same as in the previous part.

1.31 *[This problem depends on Problem 30.]* In Problem 30 you looked at two objects: Object A follows the differential equation $d^2x/dt^2 = x$, and Object B follows $d^2x/dt^2 = -x$.

(a) Solve the differential equation for Object A "by inspection": that is, think of a function $x(t)$ that is its own second derivative. (Do not use $x(t) = 0$.) Does this function match the behavior you predicted? (It may or may not, depending on the function you choose.)
Solution: One function that is its own second derivative is $x = e^t$. It does indeed head toward infinity, as we predicted Object A would. You could use $x = e^{-t}$ instead, which does not match the described behavior.
If you look more closely, you may notice that neither answer perfectly replicates the behavior described in the problem. Object A, in all the given scenarios, is supposed to start at rest: that is, $x'(0) = 0$. You do not get that behavior from either $x = e^t$ or $x = e^{-t}$. You do get it from $x = e^t + e^{-t}$ which still solves the differential equation. Later sections will discuss finding such solutions.
Final Answer: e^t or e^{-t}

(b) Solve the differential equation for Object B "by inspection": that is, think of a non-zero function $x(t)$ that is *negative* its own second derivative. (*Hint*: just throwing a negative into your answer for Object A doesn't work. Try it!) Does this function match the behavior you predicted? (It should.)
Solution: There are two functions that are negative their own second derivative: $x = \sin t$ and $x = \cos t$. Since both are periodically oscillating functions, they do indeed match the behavior you predicted earlier for Object B. The preferred solution in this case is $x = \cos t$ since that also meets the initial condition $x'(t) = 0$.
Final Answer: $\cos t$ or $\sin t$

(c) Object C has $k/m = 9$. Solve its differential equation by inspection. When k/m is positive, what effect does its value have on the behavior of the system?
Solution: Differentiating $x = e^{3t}$ brings out a factor of 3, so differentiating it twice gives you a factor of 9. Thus, $x = e^{3t}$ is a solution. In general, when k/m is positive, one solution will be $x = e^{\sqrt{k/m}\,t}$. The bigger the value of k/m, the bigger the value of the exponent, so the faster the exponential function increases to infinity. Therefore, when k/m is positive, its value determines how fast the object rolls off into positive or negative infinity.
Final Answer: e^{3t} or e^{-3t}

(d) Finally, Object D has $k/m = -9$. Solve its differential equation by inspection. When k/m is negative, what effect does its value have on the behavior of the system?
Solution: $x = \cos(3t)$ is a solution, because $dx/dt = -3\sin(3t)$, and so $d^2x/dt^2 = -9\cos(3t)$, which is -9 times x. (The other valid solution is $x = \sin(3t)$.) In general, when k/m is negative, one solution will be $x = \cos(\sqrt{k/m}\,t)$. The more negative the value of k/m, the higher the frequency of the cosine function, so the faster the object oscillates. Thus, when k/m is negative, its value determines the frequency of oscillation of the system.
Final Answer: $x = \cos(3t)$ or $x = \sin(3t)$

1.32 A quantity changes according to the differential equation $dy/dx = y^2$. You are going to draw three possible graphs that could represent such a quantity. All three graphs will be drawn on the domain $0 \le x \le 1$.

(a) The first graph starts at the point $(0, 0)$. Our differential equation promises us that the slope at that point is zero—the graph is horizontal—so move horizontally to the right (just a small distance) from that point.
Solution: See below.

(b) At the new point you reach, what is the slope, according to the differential equation? Based on that new slope, draw a little more of the graph.
Solution: Since moving horizontally doesn't change the value of y, the slope is still 0, so the graph is still horizontal. (See below.)
Final Answer: 0

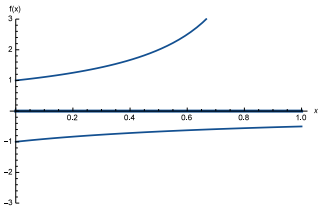
(c) Keep going in this way to draw one possible graph.
Solution: See below.

(d) Now start over at the point $(0, 1)$. At this point, our differential equation promises us a slope of 1, so move up-and-right a bit at a 45° angle.
Solution: See below.

(e) At the new point you reach, is the slope the same, higher, or lower than 1? Based on that new slope, draw a bit more of the graph.
Solution: Because y and the slope is y^2, the slope will also get higher. See below.
Final Answer: higher than 1

(f) Keep going in this way to draw a second possible graph.
Solution: See below.

(g) Now draw a third graph in the same way, starting at the point $(0, -1)$.
Solution: At $(0, -1)$, the differential equation gives you a slope of 1, so move up-and-right a bit at a 45^o angle. The new point you reach will be a negative fraction, perhaps somewhere around $y = -1/2$ or $y = -3/4$. Taking this negative fraction and putting it into the DE gives you a slope that is a positive fraction less than 1. So move up-and-right at an angle less steep than 45^o. Keep repeating this, and you will see that the graph of y will be increasing towards $y = 0$ at less and less steeper angles (concave down shape).



(h) For each of the three graphs you drew, is y increasing, decreasing, or staying constant?
Final Answer: staying constant, increasing, increasing

You will explore this equation further in Section 1.9 Problem 1.198 (see felderbooks.com).

1.33 A "simple harmonic oscillator" (SHO) is any system that obeys a differential equation of the form $d^2x/dt^2 = -\omega^2 x$ where ω is a constant. The simplest possible SHO is one for which $d^2x/dt^2 = -x$.

(a) Show that $x = \sin t$ and $x = \cos t$ are both solutions to $d^2x/dt^2 = -x$.
Solution: Differentiating $x = \sin t$ twice will give you $d^2x/dt^2 = -\sin t$, and differentiating $x = \cos t$ twice will produce $d^2x/dt^2 = -\cos t$. Both functions satisfy the differential equation, so both functions are solutions.

(b) Each of these solutions represents an oscillation. The "period" of an oscillation is defined as the time it takes to complete one full cycle from its maximum value to its minimum value and back to the maximum again. What is the period of the oscillations described by $d^2x/dt^2 = -x$?
Final Answer: 2π

(c) If one object has position $x = \sin t$ and a different object $x = \cos t$, we have just seen that they obey the same differential equation and oscillate with the same period. If you observed the motion of both objects, how would they differ?
Solution: The motion is identical except they start at different positions, so they are out of phase with each other.
Final Answer: phase

(d) Now suppose an SHO is described by $d^2x/dt^2 = -4x$. Write a solution to this equation and find its period.
Solution: $x = \sin(2t)$ is a solution, because it leads to $dx/dt = 2\cos(2t)$, and so $d^2x/dt^2 = -4\sin(2t)$, which is -4 times x. Its period is π. (The other valid solution is $x = \cos(2t)$, which has the same period.)
Final Answer: $x = \sin(2t)$ or $x = \cos(2t)$, Period = π

(e) What is the period of an SHO that obeys $d^2x/dt^2 = -\omega^2 x$?
Solution: $x = \sin(\omega t)$ and $x = \cos(\omega t)$ are both valid solutions, and both have period $2\pi/\omega$.
Final Answer: $2\pi/\omega$

(f) What is the period of a 2 kg mass oscillating on a spring with spring constant $k = 200$ N/m? (Recall that the spring exerts a force $F = -kx$ on the mass.)
Solution: The differential equation for a mass m on a spring is $d^2x/dt^2 = -(k/m)x$. This is a SHO with $\omega = \sqrt{k/m}$. Since the period of a SHO is $2\pi/\omega$, the period of the mass on a spring is $2\pi/(\sqrt{k/m}) = 2\pi\sqrt{m/k} = 2\pi\sqrt{(2kg)/(200N/m)} = \pi/5$ seconds.
Final Answer: $\pi/5$ seconds

1.34 Constrained Population Growth

The *logistic equation* is used to model population growth in a constrained environment. As an example, the differential equation $dN/dt = (0.15)N\,(1 - N/7500)$ has been used to model the elephant population in South Africa's Kruger National Park.[1] $N(t)$ is the number of elephants and t is measured in years.

(a) According to this model, what is dN/dt when $N = 0$? Explain what this result means in terms of the elephant population.

Solution: $dN/dt = 0$, so the elephant population will stay at 0. The birth rate and the death rate will both be 0 if there are no elephants to start with.

Final Answer: $dN/dt = 0$, so the elephant population will stay at 0.

(b) According to this model, what is dN/dt when $N = 7500$? Explain what this result means in terms of the elephant population.

Solution: $dN/dt = 0$, so the elephant population will stay at $N = 7500$. The number of elephants that are born each year is equal to the number of elephants that die each year, so the total number of elephants isn't changing.

Final Answer: $dN/dt = 0$, so the elephant population will stay at $N = 7500$.

(c) According to this model, what happens to dN/dt when $N > 7500$? Explain what this result means in terms of the elephant population.

Solution: $(1 - N/7500) < 0$, so $dN/dt < 0$. The elephant population decreases.

Final Answer: $dN/dt < 0$, so the elephant population decreases

(d) Make a table of N and dN/dt. Include $N = 10$, $N = 1000$, $N = 2000$, $N = 3000$, $N = 4000$, $N = 5000$, $N = 6000$, and $N = 7000$, plus the values you already calculated at $N = 0$ and $N = 7500$.

Solution:

$N = 0$	$N'(t) = 0$
$N = 10$	$N'(t) \approx 1$
$N = 1000$	$N'(t) = 130$
$N = 2000$	$N'(t) = 220$
$N = 3000$	$N'(t) = 270$
$N = 4000$	$N'(t) = 280$
$N = 5000$	$N'(t) = 250$
$N = 6000$	$N'(t) = 180$
$N = 7000$	$N'(t) = 70$
$N = 7500$	$N'(t) = 0$

(e) Calculate the exact N-value at which the elephant population would grow the fastest. In other words, calculate the N-value that maximizes dN/dt.

Solution: To find the value of N where $(0.15)N\,(1 - N/7500)$ has a maximum, we take its derivative with respect to N and set that equal to zero.

$$0.15\left(1 - \frac{N}{7500}\right) - 0.15N\frac{1}{7500} = 0$$

This can easily be solved to give $N = 3750$. (You could check to verify that this is a maximum, but you can easily tell that it is looking at the table you just made.)

Final Answer: 3750

(f) The initial condition in this case is that, in 1905 (which we might as well call $t = 0$), the population was $N = 10$. Based on your answers above and this initial condition, describe qualitatively the kind of growth you would expect to see over time. Where would the population head over time?

[1] Gallego, Samy (2003). *Modelling Population Dynamics of Elephants*. PhD thesis, Life and Environmental Sciences, University of Natal, Durban, South Africa.

Solution: The slope at the point (0, 10) is approximately 1, so the population starts out increasing slowly. As the population rises, the rate of growth increases as well, so the graph continues to rise even more steeply with time. After the population passes 3750, the population will still grow, but the growth rate will start to decrease, so the graph will be rising less steeply with time. As the population gets closer to 7500, the growth rate gets closer to 0, so the graph begins to level off. The population will keep getting closer to 7500 but never reach it.

Final Answer: 7500

(g)  The solution to this differential equation and initial condition is $N(t) = 7500e^{0.15t}/(e^{0.15t} + 749)$. Graph this function for $0 \le t \le 100$ and make sure it matches your qualitative prediction from Part f. (If it doesn't, figure out what went wrong!)

1.35 Exploration: Compound Interest (by hand)

In the Explanation (Section 1.2.2), Shannon put \$1000 into a bank account with 5% interest. Modeling her account with the equation $dM/dt = M/20$, we found that she earned slightly more than $\$1000 \times 5\% = \$50$ in her first year. Her balance is not incremented once a year, but incremented *continuously,* so she is always getting interest on the interest she just got. This distinction can be approached rigorously through a limit.

(a) Begin with an equation that literally models a 5% increase once a year. If Shannon begins the year with $\$M_0$, then after one year, she has $\$1.05M_0$. The following year she has $1.05 \times \$1.05M_0 = \$1.05^2M_0$. How much money does she have after t years?
Final Answer: $M(t) = (\$1.05)^tM_0$

(b) Now, suppose Shannon's interest is "compounded" ten times per year. This means that after each 1/10 of the year, she receives 1/10 of 5% interest (her money multiplies by 1.005). If she receives such interest ten times per year, how much does she have after one year? After t years?
Final Answer: $M(1) = (\$1.005)^{10}M_0$, $M(t) = (\$1.005)^{10t}M_0$

(c) The above thinking leads to the formula for compound interest. If Shannon starts with M_0 dollars and receives 5% interest compounded n times per year, then after t years she ends up with $M_0(1 + .05/n)^{nt}$ dollars. To compound *continuously,* take the limit of this formula as $n \to \infty$. (The most common trick for taking this limit starts by taking the natural log of the formula, using the laws of logs and l'Hôpital's rule to find the limit, and then raising e to your answer at the end.)
Solution:

$$\ln M = \ln M_0 + nt \ln(1 + .05/n)$$
$$\ln M = \ln M_0 + \frac{t \ln(1 + .05/n)}{1/n}$$

We took that unusual-looking last step, turning n in the numerator into $1/n$ in the denominator, because now in the limit this limit looks like $0/0$. We can therefore use l'Hôpital's Rule in the first step below. (Remember that we are taking a limit with respect to n, so we take derivatives with respect to n, treating t as a constant.) The step after that is just algebra.

$$\lim_{n\to\infty} \frac{t \ln(1 + .05/n)}{1/n} = \lim_{n\to\infty} \frac{t/(1 + .05/n) \times (-.05/n^2)}{-1/n^2} = \lim_{n\to\infty} \frac{.05t}{1 + .05/n} = .05t$$

Now that we know what the fraction is approaching, we can find $M(t)$.

$$\ln M = \ln M_0 + 0.05t$$
$$M = e^{\ln M_0 + 0.05t}$$
$$M = M_0 e^{0.05t}$$

Final Answer: $M = M_0e^{.05t}$

(d) Show that the answer to Part c is a valid solution to the differential equation $dM/dt = M/20$ with the initial condition $M(0) = M_0$.
Solution: You can see that it satisfies the initial condition by plugging in $t = 0$, which gives $M = M_0$. To check that it satisfies the differential equation take the derivative $dM/dt = .05M_0e^{.05t}$, which does equal the original function divided by 20.

(e) Now apply the same logic to a population growth scenario. If *every rabbit produces on average ten new rabbits per year,* then a starting population of 15 rabbits will produce 150 rabbits in their first year. What is $R(t)$ in this case? (Make sure your function predicts $R(1) = 15 + 150 = 165$.) If every rabbit produces on average 1 new rabbit every tenth

of a year, what is $R(t)$? If every rabbit produces on average $10/n$ rabbits n times per year, what is $R(t)$? Take the limit of your answer as $n \to \infty$ and show that it satisfies the differential equation $dR/dt = 10R$ and the initial condition $R(0) = 15$.

Solution: If every rabbit produces 10 new rabbits per year then each year the rabbit population is multiplied by 11, so

$$R(t) = 15(11)^t$$

If every rabbit produces 1 new rabbit each tenth of a year, then each year the rabbit population is multiplied by 2 ten times, so

$$R(t) = 15(2)^{10t}$$

If every rabbit produces $10/n$ new rabbits each $1/n$th of a year, then each year the rabbit population is multiplied by $(1 + 10/n)$ a total of n times, so

$$R(t) = 15(1 + 10/n)^{nt}$$

Following the same steps as above

$$\lim_{n\to\infty} \ln(R) = 15 + \lim_{n\to\infty} \frac{\ln(1 + 10/n)}{1/(nt)} = 15 + \lim_{n\to\infty} \frac{10t}{1 + 10/n} = 15 + 10t$$

$$\lim_{n\to\infty} R = 15e^{10t}$$

Plugging in $t = 0$ gives $R = 15$, and the derivative $R'(t) = 150e^{10t}$ satisfies the differential equation $dR/dt = 10R$.

Final Answer: $R = 15e^{10t}$

(f) Based on your answer to Part (e), how long will it be until the Earth is covered with rabbits? Include whatever estimates you use or quantities you look up as part of your solution.

Solution: A lot depends on how you define "covered with rabbits." We chose the following definition: there are enough rabbits to completely blanket all land.

The land of Earth covers roughly $150,000,000\ \text{km}^2$. A rabbit might cover an area of roughly $200\ \text{cm}^2$. So it would take roughly 7.5×10^{15} rabbits. Our exponential formula would reach this number in between 3 and 4 years.

1.36 **Exploration: Compound Interest (by computer)**

In the Explanation (Section 1.2.2), Shannon put \$1000 into a bank account with 5% interest. We solved the equation $dM/dt = M/20$ and got $M(t) = (\$1000)e^{t/20}$, and then noted that this causes her money to increase by more than 5% after a year because her balance keeps growing, and she immediately starts earning interest on that new balance. In this problem we take a numerical approach to the question: how much money does Shannon have after 20 years? In all cases, round off your answer to the nearest penny.

(a) If Shannon's money is "compounded annually"—that is, if she receives exactly 5% of her balance once a year—then the formula for her balance is $M(t) = 1000 \times 1.05^t$. Compute her balance after 20 years.
Final Answer: \$2,653.30

(b) If Shannon's money is "compounded monthly"—that is, if she receives 1/12 of 5% of her balance every month—then the formula for her balance is $M(t) = 1000 \times (1 + .05/12)^{12t}$. Compute her balance after 20 years.
Final Answer: \$2,712.64

(c) If Shannon's money is "compounded daily"—that is, if she receives 1/365 of 5% of her balance every day—then the formula for her balance is $M(t) = 1000 \times (1 + .05/365)^{365t}$. (Ignore leap years.) Compute her balance after 20 years.
Final Answer: \$2,718.10

(d) Many banks advertise that your money is "compounded continuously." Use a computer to take the limit of the above process as $n \to \infty$.
Solution: Plug in bigger and bigger values of n into $M = 1000 \times (1 + .05/n)^{n20}$ until the balance stops increasing to the nearest penny. (Of course, the balance will still continue to increase after that point, but this increase will affect only the values that are smaller than a penny.)

$$\begin{array}{ll} n = 1000 & M = \$2,718.21 \\ n = 2000 & M = \$2,718.25 \\ n = 10,000 & M = \$2,718.28 \\ n = 20,000 & M = \$2,718.28 \\ n = 100,000 & M = \$2,718.28 \end{array}$$

Final Answer: \$2,718.28

(e) Plug $t = 20$ into the solution $M(t) = (\$1000)e^{t/20}$. Comparing the result to your answers above, what kind of compounding does this solution represent?
Solution: \$2,718.28, same as (d), so it represents continuous compounding.
Final Answer: \$2,718.28, continuous

1.37 Consider the linear differential equation $dy/dx = 3 - y$.

(a) For the function $y = e^{-x} + 3$, calculate dy/dx.
Final Answer: $-e^{-x}$

(b) For the function $y = e^{-x} + 3$, calculate and simplify $3 - y$.
Final Answer: $-e^{-x}$

(c) Your answers to Parts a and b should have come out the same, demonstrating that $y = e^{-x} + 3$ is a solution to this differential equation. Using the same method, demonstrate that $y = 7e^{-x} + 3$ is *also* a solution to this differential equation.
Solution: $dy/dx = -7e^{-x}$ and $3 - y = 3 - (7e^{-x} + 3) = -7e^{-x}$, so both sides of the differential equation come out the same, demonstrating that $y = 7e^{-x} + 3$ is indeed a solution.

(d) Show that the function $y = Ce^{-x} + 3$ is a solution to this differential equation for any value of the constant C. (You do this the same way you did the previous problems: just let C work through the equations as a constant.)
Solution: $dy/dx = -Ce^{-x}$ and $3 - y = 3 - (Ce^{-x} + 3) = -Ce^{-x}$, so both sides of the differential equation come out the same for any value of C.

(e) What specific function does this general solution give you for the value $C = 0$? Show that this function is a solution to the differential equation.
Solution: It gives you the specific function $y = 3$. Since $dy/dx = 0$ and $3 - y = 0$, $y = 3$ is a solution to the differential equation.
Final Answer: $y = 3$

(f) Find the one and only function that satisfies both the differential equation $dy/dx = 3 - y$ and the initial condition $y(0) = 1$.
Solution: Plugging $x = 0$ and $y = 1$ into the general solution, $1 = Ce^{-0} + 3$ solves to give $C = -2$. So $y = -2e^{-x} + 3$ is the function that satisfies both the differential equation and the initial condition.
Final Answer: $y = -2e^{-x} + 3$

(g) Show that the function $y = 3 - e^{10-x}$ is a solution to this differential equation. Then explain why this does not violate the rule that the solution in Part d is the general solution representing all possible solutions.
Solution: $dy/dx = e^{10-x}$ and $3 - y = e^{10-x}$. They come out the same so this is a solution. But this solution can also be written in the form $y = -e^{10}e^{-x} + 3$, fitting the form of the general solution given previously with $C = -e^{10}$.

1.39 $P'(t) = -10P(t)$
Solution: $P(t) = Ce^{-10t}$
Condition: $P(0) = 7$

Solution: (a) $P'(t) = -10Ce^{-10t} - 10P(t) = -10Ce^{-10t}$
(b) $P(0) = Ce^0 = 7$ solves to give $C = 7$. Thus, $P(t) = 7e^{-10t}$.
Final Answer: $P(t) = 7e^{-10t}$

1.40 $u'(t) = \sqrt{u}$

Solution: $u(t) = (t/2 + C)^2$

Condition: $u(0) = 5$

Solution: (a) $u'(t) = 2(t/2 + C) \times 1/2 = t/2 + C$

$\sqrt{u} = \sqrt{(t/2 + C)^2} = t/2 + C$

(b) $u(0) = C^2 = 5$ gives you two possible values, $C = \pm\sqrt{5}$. Therefore, there are two particular solutions, $u(t) = (t/2 \pm \sqrt{5})^2$, which match the given initial condition.

Final Answer: $u(t) = (t/2 \pm \sqrt{5})^2$

1.41 $\sqrt{x}\dfrac{dy}{dx} = y^2 e^{\sqrt{x}}$

Solution: $y = \dfrac{-1}{2e^{\sqrt{x}} + C}$

Condition: $y(0) = 5$

Solution: (a)

$$y = -(2e^{\sqrt{x}} + C)^{-1}$$
$$\frac{dy}{dx} = (2e^{\sqrt{x}} + C)^{-2} \times 2e^{\sqrt{x}} \times \frac{1}{2}x^{-\frac{1}{2}}$$
$$\sqrt{x}\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{(2e^{\sqrt{x}} + C)^2}$$
$$y^2 e^{\sqrt{x}} = \frac{1}{(2e^{\sqrt{x}} + C)^2} \times e^{\sqrt{x}}$$

(b) $y = \dfrac{-1}{2e^{\sqrt{0}} + C} = 5$ solves to give $C = -\dfrac{11}{5}$. Thus, $y = \dfrac{-1}{2e^{\sqrt{x}} - \frac{11}{5}}$.

Final Answer: $y = -1/(2e^{\sqrt{x}} - 11/5)$

1.42 $dy/dx + y/x = 4 \ln x \ (x > 0)$
Solution: $y = 2x \ln x - x + C/x$
Condition: $y(1) = 10$

Solution: (a) $dy/dx = 2 \ln x + 2 - 1 - C/x^2 = 2 \ln x + 1 - C/x^2$
$y/x = 2 \ln x - 1 + C/x^2$
$dy/dx + y/x = 4 \ln x$
(b) $y = 2(1) \ln 1 - 1 + C/1 = 10$ solves to give $C = 11$. Thus, $y = 2x \ln x - x + 11/x$.

Final Answer: $y = 2x \ln x - x + 11/x$

1.43 $dy/dx = (1 - 2xy)(x^2 + 1)$
Solution: $y = (C + x)(x^2 + 1)$
Condition: $y(1) = -1$
Solution: (a)

$$\frac{dy}{dx} = \frac{(1)(x^2+1) - (C+x)(2x)}{(x^2+1)^2}$$

$$\frac{1-2xy}{x^2+1} = \frac{\frac{x^2+1}{x^2+1} - \frac{2x(C+x)}{x^2+1}}{x^2+1}$$

$$= \frac{(x^2+1) - (C+x)(2x)}{(x^2+1)^2}$$

(b) $y = (C + 1)(1^2 + 1) = -1$ solves to give $C = -3/2$. Thus, $y = (x - 3/2)(x^2 + 1)$.
Final Answer: $y = (x - 3/2)(x^2 + 1)$

1.44 $d^2x/dt^2 = -9.8$

Solution: $x = C_1 + C_2t - 4.9t^2$

Conditions: $x(0) = 6, x'(0) = -3$

Solution: (a) $dx/dt = C_2 - 4.9(2)t = C_2 - 9.8t$

$d^2x/dt^2 = -9.8$

(You may recognize this DE as the acceleration due to Earth's gravity and the solution as the kinematic equation for position in the vertical direction for a projectile with an initial vertical position of C_1 and an initial vertical velocity C_2.)

(b) $x(0) = C_1 + C_2(0) - 4.9(0)^2 = 6$ solves to give $C_1 = 6$

$x'(0) = C_2 - 9.8(0) = -3$ solves to give $C_2 = -3$

Thus, $x = 6 - 3t - 4.9t^2$ is the particular solution.

Final Answer: $x = 6 - 3t - 4.9t^2$

1.45 $u''(t) + 4t^3 = 4u(t) + 6t$

Solution: $u(t) = t^3 + C_1 e^{2t} + C_2 e^{-2t}$

Conditions: $u(0) = 5$, $u'(0) = 6$

Solution: (a) $u'(t) = 3t^2 + 2C_1 e^{2t} - 2C_2 e^{-2t}$

$u''(t) = 6t + 4C_1 e^{2t} + 4C_2 e^{-2t}$

$u''(t) + 4t^3 = 6t + 4C_1 e^{2t} + 4C_2 e^{-2t} + 4t^3$

$4u(t) + 6t = 4t^3 + 4C_1 e^{2t} + 4C_2 e^{-2t} + 6t$

(b) $u(0) = C_1 + C_2 = 5$

$u'(0) = 2C_1 - 2C_2 = 6$

Solving this system of equations gives you $C_1 = 4$ and $C_2 = 1$. Thus, $u(t) = t^3 + 4e^{2t} + e^{-2t}$.

Final Answer: (b) $u(t) = t^3 + 4e^{2t} + e^{-2t}$

1.46 For each scenario write a differential equation to describe the situation, specify how many arbitrary constants would be in the general solution, and give one possible set of physical conditions that could be used to find the values of those arbitrary constants. For example, for a mass on a spring that exerts a force $F = -kx$ you might write the following:

- $m(d^2x/dt^2) = -kx$
- Since this is a second-order equation the general solution will have two arbitrary constants.
- To specify their values you could use the position and velocity of the mass at $t = 0$.

(a) A population of mice $M(t)$. Each year every mouse has on average 4 babies, and predators kill 100 of the mice.

Solution: Since $dM/dt = 4M - 100$ is a first-order linear DE, the general solution will have one arbitrary constant. To find this constant, you could use the population of mice at $t = 0$.

Final Answer: $dM/dt = 4M - 100$, one arbitrary constant, population at $t = 0$.

(b) A sample of radioactive material $R(t)$. In some fixed interval of time half of the sample decays. (Since the half-life isn't specified here your differential equation will have an unknown constant in it, separate from the arbitrary constants that would appear in the solution if you were to solve it.)

Solution: In radioactive decay, the rate of decay is proportional to the amount of radioactive material present. Hence, $dR/dt = -kR$, where k is the unknown decay constant. Since this is a first-order linear DE, the general solution will have one arbitrary constant. To find this constant, you could use the amount of radioactive material at $t = 0$.

Final Answer: $dR/dt = -kR$, one arbitrary constant, amount of material at $t = 0$

(c) The temperature in a long hallway $T(x)$ obeys the differential equation $d^2T/dx^2 = 0$. (You don't have to write the differential equation for this one since we gave it to you.)

Solution: Since this is a second-order linear DE, there are two arbitrary constants in the general solution. To find these, you could use the temperature reading at the two ends of the hallway. (Since it is much easier to measure temperature than to measure the first derivative of temperature, it will be easier for you to take two temperature readings rather than one temperature and one first derivative reading at any one point.)

Final Answer: two arbitrary constants, temperature at the two ends of the hallway

(d) A falling object experiences a constant gravitational force $-mg$ and an upward drag force proportional to the object's speed.

Solution: Such a problem is based on Newton's second law $F_{\text{total}} = ma$. The a in that equation is d^2x/dt^2 so this is a second-order differential equation. In this case there are two forces acting. One is the constant force $-mg$ of gravity. The other is the drag force, which we can write as $F_{drag} = -bv$ where b is a positive constant. (The negative sign indicates that the force always points in the opposite direction of the velocity, which is what drag forces do.)

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - mg$$

Such a second-order differential equation would require two initial conditions and therefore two arbitrary constants. The most likely conditions would be the position and velocity at one particular moment, which we might designate as $x(0)$ and $v(0)$.

Final Answer: $d^2x/dt^2 = -(b/m)(dx/dt) - g$, two arbitrary constants, position and velocity at $t = 0$

1.47 Consider the differential equation $x(dy/dx) = y \ln y$.

(a) Many differential equations can be solved for any given initial conditions. This equation, however, is restricted. For instance, we can see immediately that no solution is possible with the initial condition $y(1) = -1$. Why? Another impossible condition is $y(0) = 2$. Why? (Assume all variables are real.)
Solution: The presence of $\ln y$ on the right side of the differential equation imposes the limitation $y > 0$, which rules out $y(1) = -1$.
If $x = 0$ then the left side is zero, which means the right side must be zero, which means y must be 1, which rules out $y(0) = 2$.

(b) One solution to this problem is the constant function $y = 1$. Show that this function solves the equation by finding $x(dy/dx)$ and $y \ln y$ and showing they are the same.
Solution: $x(dy/dx)$ and $y \ln y$ both equal zero in this case

(c) The general solution to this equation is $y = e^{Cx}$. Show that this function solves the equation by finding $x(dy/dx)$ and $y \ln y$ and showing they are the same.
Solution:

$$\frac{dy}{dx} = e^{Cx} \times C$$
$$x\frac{dy}{dx} = e^{Cx} \times Cx$$
$$y \ln y = e^{Cx} \times (Cx)$$

Both sides of the DE give $e^{Cx} \times Cx$.

(d) If the function given in Part c is *the general* solution, it should include all solutions—including the one in Part b. Show that it does by finding the value of C that leads to that solution.
Final Answer: $C = 0$

(e) Of all the functions that solve this differential equation, only one of them has the property that $y(2) = 5$. By plugging $x = 2$ and $y = 5$ into the general solution, find the value of C that matches this condition.
Final Answer: $C = (1/2) \ln 5$

1.48 For the differential equation $dy/dx = 1 - y^2$, the general solution is $y = (Ce^{2x} - 1)/(Ce^{2x} + 1)$.

(a) List all of the following functions that are valid solutions to this differential equation. (You do *not* have to test each one out in the differential equation, which would be tedious. Instead, see which functions fit the "template" provided by the general solution.)

i. $y = 0$
ii. $y = -1$
iii. $y = e^{2x}$
iv. $y = (3e^{2x} - 1)/(3e^{2x} + 1)$
v. $y = (-5e^{2x} - 1)/(-5e^{2x} + 1)$
vi. $y = (3e^{2x} - 1)/(-5e^{2x} + 1)$

Final Answer: (ii), (iv), and (v)

(b) Choose *one* of the functions that you said was a valid solution, and demonstrate that it does indeed satisfy the differential equation. (Go ahead, choose the easy one!)

Solution: For $y = -1$, dy/dx and $1 - y^2$ are both 0.

(c) Find the function that satisfies this differential equation with the initial condition $y(0) = 5$. (It is not one of the functions listed above.)

Solution: Plug in $x = 0$ and $y = 5$ into the general solution to find that $C = -3/2$ for this initial condition.

$$y = \frac{-(3/2)e^{2x} - 1}{-(3/2)e^{2x} + 1}$$

Final Answer: $y = \dfrac{-(3/2)e^{2x} - 1}{-(3/2)e^{2x} + 1}$

(d) Find $\lim_{C\to\infty}$ of the general solution given in the problem and show that the resulting function is also a valid solution.

Solution: In the limit $C \to \infty$, both the numerator and the denominator of the general solution approach infinity. You can use l'Hôpital's Rule to take the derivative with respect to C of both the top and bottom:

$$\lim_{C\to\infty} \frac{\frac{d}{dC}(Ce^{2x} - 1)}{\frac{d}{dC}(Ce^{2x} + 1)} = \lim_{C\to\infty} \frac{e^{2x}}{e^{2x}} = 1$$

Thus, the $\lim_{C\to\infty}$ of the general solution is equal to 1. The resulting function $y = 1$ is a valid solution because both dy/dx and $1 - y^2$ are 0.

Final Answer: 1

1.49 Two particular solutions of the equation $4f''(z) + f(z) = 4f'(z)$ are $f_1(z) = e^{z/2}$ and $f_2(z) = ze^{z/2}$.

(a) Show that any linear combination $Af_1(z) + Bf_2(z)$ is also a valid solution to the differential equation.

Solution: Let $f = Af_1 + Bf_2$

$f' = Af_1' + Bf_2'$

$f'' = Af_1'' + Bf_2''$

$4f'' + f = 4(Af_1'' + Bf_2'') + (Af_1 + Bf_2) = A(4f_1'' + f_1) + B(4f_2'' + f_2) = A(4f_1') + B(4f_2')$

In the last step, use the fact that f_1 and f_2 are solutions to the differential equation to rewrite the parts in the parentheses.

$4f' = 4(Af_1' + Bf_2') = 4Af_1' + 4Bf_2'$

Since both sides of the differential equation come out the same, then f is a solution.

(b) Find the particular solution that satisfies the conditions $f(2) = 14e$ and $f'(2) = 12e$.

Solution: Plugging $z = 2$ into the general solution $f(z) = Ae^{z/2} + Bze^{z/2}$ and setting that equal to $14e$ gives you:

(1) $Ae + 2Be = 14e$

The derivative of the general solution is $f'(z) = (1/2)Ae^{z/2} + B[e^{z/2} + z(1/2)e^{z/2}]$.

Plugging $z = 2$ into $f'(z)$ and setting that equal to $12e$ gives you:

(2) $(1/2)Ae + 2Be = 12e$

Solving (1) and (2) simultaneously, you get $A = 4$ and $B = 5$. Therefore, $f(z) = 4e^{z/2} + 5ze^{z/2}$ is the particular solution that satisfies those initial conditions.

Final Answer: $f(z) = 4e^{z/2} + 5ze^{z/2}$

1.50 A simple harmonic oscillator follows the differential equation $d^2x/dt^2 = -9x$.

(a) Show that the function $x = C\sin(3t)$ is a valid solution of this equation for any value of the constant C.
Solution: For this function $x'(t) = 3C\cos(3t)$ so $x''(t) = -9C\sin(3t)$ which is indeed $-9x(t)$.

(b) Show that it is impossible, using the solution from Part a, to meet the initial conditions $x(0) = 4$ and $x'(0) = 12$.
Solution: Plugging $t = 0$ into $x = C\sin(3t)$ gives you $x = 0$, for any value of C you choose. Therefore, you can never make $x = 4$ if you use the solution in this form.

(c) Show that $x = C\sin(3t + \phi)$ is a valid solution to the differential equation for any values of the constants C and ϕ.
Solution:

$$dx/dt = 3C\cos(3t + \phi)$$
$$d^2x/dt^2 = -9C\sin(3t + \phi) = -9x$$

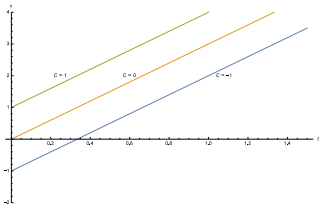
(d) Find the function that solves this differential equation and meets the initial conditions given in Part b.
Solution: Plugging $t = 0$ and $x = 4$ into $x = C\sin(3t + \phi)$ gives you:
(1) $4 = C\sin\phi$.
Plugging $t = 0$ and $x' = 12$ into $x'(t) = 3C\cos(3t + \phi)$ gives you:
(2) $12 = 3C\cos\phi$.
Dividing (1) by (2), you get: $1 = \tan\phi$. This tells you that $\phi = \pi/4$. Plugging $\phi = \pi/4$ into either (1) or (2) will give you $C = 4\sqrt{2}$.
Therefore, the particular solution that satisfies those initial conditions is: $x = 4\sqrt{2}\sin(3t + \pi/4)$.
Final Answer: $x = 4\sqrt{2}\sin(3t + \pi/4)$

1.51 One of the simplest differential equations is the equation that says "The quantity x grows at a constant rate." Suppose that every one unit of time t, the quantity x grows by exactly k.

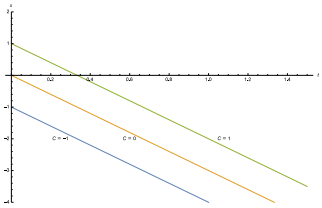
(a) Write a differential equation that expresses this rule.
Final Answer: $dx/dt = k$

(b) Write the general solution $x(t)$ to your differential equation. Note that your solution will have two unrelated constants in it: the k from the equation, and the C from the solution.
Final Answer: $x = kt + C$

(c) Choosing $k = 3$, graph the different solutions for three different C-values.
Solution:



(d) Choosing $k = -3$, graph the solutions for the same three C-values.
Solution:



(e) Explain in words: what kind of growth results from this differential equation? How does k affect the behavior of the system and how does C affect it?
Solution: A linear growth; k tells you the rate of growth while C tells you how much you started with.

1.52 A simple model for population growth is "the more you have, the more you get": put more mathematically, "the rate of growth of the population is proportional to the population itself."

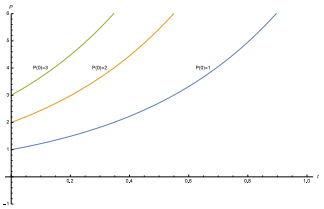
(a) Write a differential equation that expresses this model. Use P for the population, t for the time, and k for the constant of proportionality.
Final Answer: $dP/dt = kP$

(b) Solve your differential equation "by inspection": that is, look at it until you can think of any function that works. (For the moment, avoid the "trivial" solution $P(t) = 0$. We'll come back to that one.)
Final Answer: $P = e^{kt}$

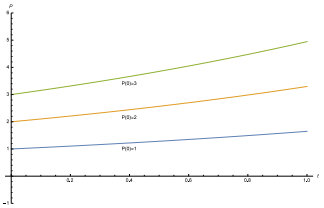
(c) Now, start to modify your solution. If you add 5, does it still work? If you multiply by 3, does it still work? Your goal here is to arrive at the general solution: that is, a function that satisfies the original differential equation for any value of an arbitrary constant C.
Solution: Adding numbers to the above solution will not work, but multiplying the solution by any constant does work. If you let the general solution be $P = Ce^{kt}$, then $dP/dt = kCe^{kt} = kP$, so this solution will work for any value of an arbitrary constant.
Final Answer: $P = Ce^{kt}$

(d) The solution $P(t) = 0$ should be a special case of your general solution. Find a value of C for which your general solution becomes $P(t) = 0$.
Final Answer: $C = 0$

(e) Choosing $k = 2$, graph three different solutions of this differential equation, using different starting values $P(0)$. Make sure your graphs fit the claim that our differential equation began with: the higher the population, the faster the growth (i.e., the higher the slope)!
Solution:



(f) Choosing $k = 1/2$, graph solutions for the same three starting values $P(0)$.
Solution:



(g) Explain in words: what kind of growth results from this differential equation? How does k affect the behavior of the system and how does C affect it?
Solution: An exponential growth; k affects how fast the system grows (by affecting both the rate of growth and how fast the rate of growth changes—the bigger the k, the bigger the growth rate and the faster the growth rate increases), while C still tells you how much you started with.

1.53 A tank contains 3 l of water. 30 g of salt are mixed evenly throughout the tank. Salt water is draining from the tank at a constant rate of 1 l/h. Let S be the amount of salt (measured in grams) remaining in the tank as a function of t (measured in hours).

(a) If nothing else is happening, write a differential equation for the function $S(t)$. Begin by asking yourself: how much salt disappears from the tank each hour?

Solution: The amount of salt that disappears equals the concentration of salt times the drainage rate.

$$\frac{30\text{ g}}{3\text{ l}} \times \frac{1\text{ l}}{\text{hour}} = 10\frac{\text{g}}{\text{hour}}$$

Since this is how much salt is disappearing every hour, the differential equation is $dS/dt = -10$.

Final Answer: $dS/dt = -10$

(b) Find the solution to your differential equation in Part a with the initial condition $S(0) = 30$.

Solution: $S(t) = -10t + C$ is the general solution. Plugging $t = 0$ and $S = 30$ into this equation gives you $C = 30$.

Final Answer: $S(t) = -10t + 30$

(c) Now assume that from the beginning fresh water is being dumped into the tank at a constant rate of 1 l/h, so the total volume of the tank is kept constant at 3 l, and the salt is always mixed evenly throughout the water. Write a different differential equation for $S(t)$ in this scenario, once again beginning with the question: how much salt disappears from the tank each hour?

Solution: Again, the amount of salt that disappears equals the concentration of salt times the drainage rate. Since the salt is leaving but the water volume is staying the same, the concentration will change with time. The concentration at any particular time is just the total amount of salt in the tank at that time divided by the total amount of water in the tank, so it is $S(t)/3$ g/liter. The drainage rate is still the same, 1 liter/hr. So the amount of salt that disappears is $S(t)/3 \times 1 = S(t)/3$.

$$\frac{dS}{dt} = -\frac{1}{3}S$$

Final Answer: $dS/dt = -S/3$

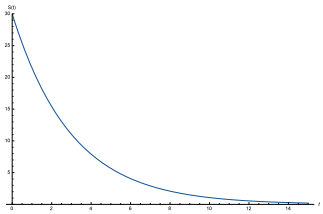
(d) Find the solution to your differential equation in Part c with the initial condition $S(0) = 30$.

Solution: $S(t) = Ce^{-(1/3)t}$ is the general solution. Plugging $t = 0$ and $S = 30$ into this general solution gives $C = 30$.

Final Answer: $S(t) = 30e^{-(1/3)t}$

(e) Draw a quick sketch of your solution.

Solution:



1.54 Water is leaking from a hole at the bottom of a tank. The height of water in the tank decreases at a rate proportional to the square root of the height.

(a) Why, physically, does the rate of leaking change over time, instead of being constant?
Solution: The water on top pushes on the water at the bottom, creating more pressure to force it through the hole. The less water, the less pressure.

(b) Using h for the height of water in the tank and t for time, write the differential equation for $h(t)$. Your equation will contain a constant of proportionality k.
Final Answer: $dh/dt = -k\sqrt{h}$

(c) Is your constant k positive or negative? How can you tell? What are the units of k?
Solution: The height h is positive, and the rate of change dh/dt is negative because the height is decreasing. If you write $dh/dt = k\sqrt{h}$ then k is negative. For this reason we prefer $dh/dt = -k\sqrt{h}$ which leads to a positive k. It doesn't change the equations in any fundamental way, but it makes it easier to interpret the answer.
Because h has units of length and dh/dt has units of length/time, k must have units of time^{-1}.
Final Answer: If $dh/dt = k\sqrt{h}$ then $k < 0$. If $dh/dt = -k\sqrt{h}$ then $k > 0$. Either way k has units of time^{-1}.

(d) Based on your differential equation, describe in words how the height of the water will change over time.
Solution: The height h will decrease. As it decreases, dh/dt will get less negative, so the rate of decrease will slow. The height will approach zero more and more gradually.

(e) Now suppose it is raining into the top of the tank at a rate of 4 in/h. Write a new differential equation for $h(t)$.
Final Answer: $dh/dt = 4 - k\sqrt{h}$

(f) Your new differential equation has one special solution, an "equilibrium solution" where the amount of water flowing in and the amount of water flowing out are perfectly balanced. What is the height of the equilibrium solution? *Hint*: you can tell by looking at your differential equation and thinking about dh/dt at equilibrium!
Solution: At equilibrium $dh/dt = 0$ so $4 - k\sqrt{h} = 0$ so $h = 16/k^2$.
Final Answer: $h = 16/k^2$

1.55 You are running a scientific experiment on the bacteria Veribadocillin. The amount of bacteria in your container is constant, and every second it produces 2 g of the chemical Situnsene. You have filters in your container that remove 2% of the Situnsene each second.

(a) Write a differential equation for the amount of Situnsene $S(t)$ in the container.
Final Answer: $dS/dt = 2 - .02S$

(b) Verify that $S(t) = e^{-.02t} + 100$ is a solution to the equation you wrote.
Solution:

$$\frac{dS}{dt} = -.02e^{-.02t}$$

$$2 - .02S = 2 - \left(.02e^{-.02t} + 2\right) = -.02e^{-.02t}$$

(c) Use trial and error to find how you can modify this solution to write the general solution. You could try adding an arbitrary constant to S, multiplying S by an arbitrary constant, or adding or multiplying arbitrary constants to different parts of S. For each thing you try you should calculate both sides of the differential equation and check whether they match for *any value* of the arbitrary constant. Once you find a solution that works, you have the general solution.
Final Answer: $S(t) = Ce^{-.02t} + 100$

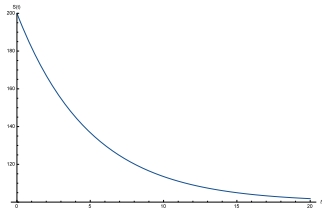
(d) Find the solution $S(t)$ if the experiment started with 200 g of Situnsene.
Solution: The initial condition gives $S(0) = 200 = C + 100$ so $C = 100$ and $S(t) = 100(e^{-.02t} + 1)$.
Final Answer: $S(t) = 100(e^{-.02t} + 1)$

(e) You should find from your equation that at late times the amount of Situnsene approaches a constant. Find the value of this constant and explain why the amount of Situnsene can stay constant at this value.
Solution:

$$\lim_{t\to\infty} S(t) = 100$$

Looking at the differential equation we see that if $S = 100$ then $dS/dt = 0$ so the amount of S never changes. Physically, if you have 100 g, then the 2% being removed every second is exactly the same as the 2 g being added every second.
Final Answer: $\lim_{t\to\infty} S(t) = 100$

(f) Draw a quick sketch of your solution.
Solution:



1.56 $dy/dx = y$ is a first-order linear differential equation.

(a) Show that $y = (C_1 + C_2)e^x$ is a solution for any constants C_1 and C_2.
Solution: $dy/dx = (C_1 + C_2)e^x = y$

(b) The general solution to a first-order linear differential equation should have only one independent arbitrary constant. Explain why the solution in Part a does not violate this rule.
Solution: $C_1 + C_2$ is just another representation for a constant, so the solution still has the form (constant)$\times e^x$. Even though C_1 and C_2 can take any values they want, their sum still gives you one constant value. So, there is only one *independent* constant.

(c) Show that $y = C_1 e^{x+C_2}$ is a solution for any constants C_1 and C_2.
Solution: $dy/dx = C_1 e^{x+C_2} = y$

(d) Explain how Part c *also* does not violate the rule that the general solution can have only one independent arbitrary constant.
Solution: $C_1 e^{x+C_2}$ can be written as $C_1 e^{C_2} e^x$, and $C_1 e^{C_2}$ is just a fancy way of saying "some constant". The solution still has the form of (constant)$\times e^x$, so there is only *one* independent constant.

(e) Write the general solution to this equation in a form containing only one arbitrary constant.
Final Answer: $y = Ce^x$

1.57 Consider the linear differential equation $d^2x/dt^2 = x$.

(a) The general solution to this equation is $x = Ae^t + Be^{-t}$, where A and B are arbitrary constants. Find the constants A and B given the initial conditions $x(0) = 0, x'(0) = 1$.
Solution: $x(0) = A + B = 0$
$x'(t) = Ae^t - Be^{-t}$
$x'(0) = A - B = 1$
Final Answer: $A = 1/2$ and $B = -1/2$

(b) An alternative way of expressing the general solution is $x = C\sinh(t) + D\cosh(t)$, where $\sinh(t) = (1/2)(e^t - e^{-t})$ and $\cosh(t) = (1/2)(e^t + e^{-t})$ and C and D are arbitrary constants. Find the constants C and D given the initial conditions $x(0) = 0, x'(0) = 1$.
Solution: $x(0) = C(1/2)(1 - 1) + D(1/2)(1 + 1) = D = 0$
$x'(t) = C(1/2)(e^t + e^{-t}) + D(1/2)(e^t - e^{-t})$
$x'(0) = C(1/2)(1 + 1) + D(1/2)(1 - 1) = C = 1$
Final Answer: $C = 1$ and $D = 0$

(c) Show that your constants in Parts a and Part b lead to the same solution (the only correct solution to this differential equation with these conditions).
Solution: Plugging in $A = 1/2$ and $B = -1/2$ into $x = Ae^t + Be^{-t}$ gives you $x = 1/2e^t - 1/2e^{-t} = (1/2)(e^t - e^{-t})$.
Plugging in $C = 1$ and $D = 0$ into $x = C\sinh(t) + D\cosh(t)$ gives you $x = \sinh(t) = (1/2)(e^t - e^{-t})$.
Both forms of the general solution reduce to the same equation when the constants specific to the initial conditions are plugged into the corresponding general solutions.

1.58 An object moves according to the simple harmonic oscillator equation $d^2x/dt^2 = -x$.

(a) Show that $x = A\sin t + B\cos t$ is a valid solution.
Solution: $dx/dt = A\cos t - B\sin t$
$d^2x/dt^2 = -A\sin t - B\cos t = -x$
So $x = A\sin t + B\cos t$ satisfies the differential equation.

(b) Show that $x = C\sin(t+\phi)$ is a valid solution.
Solution: $dx/dt = C\cos(t+\phi)$
$d^2x/dt^2 = -C\sin(t+\phi) = -x$

(c) Show that both of these general solutions are in fact the same, by expressing A and B in terms of C and ϕ or vice- versa. You may need to look up some trig identities.
Solution:

$$\begin{aligned}\sin(a+b) &= \sin a\cos b + \cos a\sin b\\ C\sin(t+\phi) &= C(\sin t\cos\phi + \cos t\sin\phi)\\ &= C\cos\phi\sin t + C\sin\phi\cos t\\ A &= C\cos\phi\\ B &= C\sin\phi\end{aligned}$$

Final Answer: $A = C\cos\phi$, $B = C\sin\phi$

1.59 A car is zooming down the road at a constant speed of 55 mph.

(a) Given that $v = 55$, and given the definition of v, write a differential equation for $x(t)$.
Final Answer: $dx/dt = 55$

(b) Solve that differential equation: that is, write some function $x(t)$ that makes that differential equation true. You may be able to write many different $x(t)$ functions that work for this. Ultimately, you want to wind up with an answer that has an arbitrary constant C in it, to represent all possible solutions.
Final Answer: $x(t) = 55t + C$

You now have an equation for $x(t)$. But it has an arbitrary constant C in it. What does that mean? Given the information we have—"a car moving at 55 mph"—we don't know where the car is, until we get more information.

(c) Here comes more information: at time $t = 0$, the car was at position $x = -5$. Use that fact to find C and write the real equation for $x(t)$: the one that will actually tell us where the car is.
Solution: Plugging in $x = -5$ and $t = 0$ into the equation for $x(t)$ above, you get $C = -5$, so $x(t) = 55t - 5$.
Final Answer: $x(t) = 55t - 5$

(d) Describe in words the motion described by your $x(t)$ function. Does it make sense with the physical situation? Why or why not?
Solution: The function starts at $x = -5$ and increases linearly with a constant slope of 55. This makes sense with the physical situation because the car starts at position $x = -5$ and moves forward with a constant speed of 55 mph.

1.60 A piano is dropped from the top of a building. Neglecting friction for the moment, the force is given by $F = -mg$, where g is a positive constant.

(a) Write and solve an algebraic equation for $a(t)$.
Solution: $F = ma = -mg$ gives you $a = -g$.
Final Answer: $a = -g$

(b) Given that, write a differential equation for $v(t)$. (The position x should not be in this equation.)
Solution: Use $a = dv/dt$ to get $dv/dt = -g$.
Final Answer: $dv/dt = -g$

(c) Solve that differential equation. Once again, you may be able to write many different $v(t)$ functions that work, so use an arbitrary constant to represent all possible solutions. Call your constant C_1 instead of just C.
Final Answer: $v(t) = -gt + C_1$

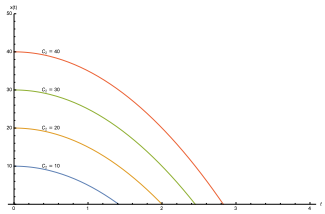
(d) Now, using your answer to Part c and the definition of velocity, write a differential equation for $x(t)$.
Final Answer: $dx/dt = -gt + C_1$

(e) Solve that. Of course, the constant C_1 will be in there, since it was part of v. But still, there will be many possible solutions. To represent them all, you will need a new arbitrary constant C_2.
Final Answer: $x(t) = -(1/2)gt^2 + C_1 t + C_2$

You now have an equation for $x(t)$ that has two arbitrary constants, C_1 and C_2. We can find the first one based on the given information.

(f) Since the piano was "dropped" (not "thrown"), we can assume that $v(0) = 0$. Use this to find the constant C_1. Then rewrite your $x(t)$ formula with only one constant, C_2.
Solution: Plugging in $v = 0$ and $t = 0$ into the equation for $v(t)$ above gives you $C_1 = 0$, so $x(t) = -(1/2)gt^2 + C_2$.
Final Answer: $x(t) = -(1/2)gt^2 + C_2$.

(g) Let's look at this graphically. Suppose $g = 10$. Pick a value for C_2 and draw a graph of your $x(t)$ function. (You can do this using a calculator or computer, but it should also be easy by hand.) Then pick a different value for C_2 and draw another graph. Continue until you have at least four graphs. What do they all have in common? How do they differ?
Solution:



All the graphs have the same downward sloping shape and curvature, but their y-intercepts (where they start) are different.

(h) To choose one specific function, we need one more piece of information that was not specified in the problem. What is it?
Final Answer: The value of C_2, which represents the initial height.

(i) Make up a reasonable answer and write the function $x(t)$ that matches it, with no arbitrary constants.
Solution: Let's say the initial height is 30 meters, so $C_2 = 30$. Therefore, for this initial condition, $x(t) = -(1/2)gt^2 + 30$.
Final Answer: $x(t) = -(1/2)gt^2 + 30$

(j) Describe the motion of the piano in words. Does it make sense with the physical situation? Does it correspond to your answer?

Solution: The function $x(t)$ starts at $x = 30$ with a horizontal slope and then decreases with progressively greater negative slopes toward negative infinity. This corresponds to the physical condition that the piano is dropped from an initial height of 30 meters and it falls down to the ground with increasing downward speeds. However, in the physical situation the piano is stopped by the pavement at $x = 0$ at some time t, whereas the function $x(t)$ keeps going down past $x = 0$ mark.

1.61 A car is rolling along a flat highway. Neither the gas pedal nor the brakes are being pushed; the only force on the car is air resistance. A simple model is to say that air resistance is proportional to velocity: that is, the faster you go, the more resistance you feel. We can write that as $F = -bv$. *Hint*: if you have trouble with this problem you may want to work through Problem 60 as a guide.

(a) What is the sign of b? Explain why this equation would not make sense physically if b had the other sign.
Solution: Air resistance pushes against the direction of motion, so the force of air resistance must be opposite that of velocity. This makes b positive.
Final Answer: positive

(b) What are the SI units of b?
Solution: From the equation $F = -bv$, b must have units of N/(m/s). Since a Newton is a kg m/s^2, b has units of kg/s.
Final Answer: kg/s

(c) Write a differential equation for v.
Solution: $F = ma = -bv$. Now use $a = dv/dt$ to get $m(dv/dt) = -bv$.
Final Answer: $m(dv/dt) = -bv$

(d) Solve it in the most general terms possible. (This will introduce an arbitrary constant.)
Solution: Rewriting the DE as $dv/dt = -(b/m)v$, you see that it has the form $dv/dt = -kv$. You already know the general solution to such an equation, which is just $v = Ce^{-kt}$. So, letting $k = (b/m)$ and $C = C_1$ (since you will have more than one constant in your final equation) the general solution is $v(t) = C_1 e^{-(b/m)t}$.
Final Answer: $v(t) = C_1 e^{-(b/m)t}$

(e) Now, use that answer to write a differential equation for x.
Final Answer: $dx/dt = C_1 e^{-(b/m)t}$

(f) Solve it in the most general terms possible. (This will introduce another constant.)
Final Answer: $x(t) = -(m/b)C_1 e^{-(b/m)t} + C_2$

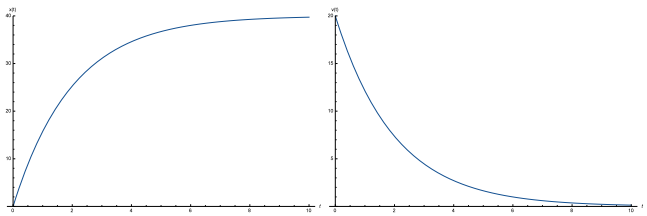
(g) Suppose the car started at position $x = x_0$ with velocity $v = v_0$. Find the values of the arbitrary constants and write the function $x(t)$ that describes the motion of the car.
Solution: Plugging $t = 0$ into the equation for $v(t)$ immediately gives $C_1 = v_0$. The equation for $x(t)$ then gives $x_0 = -(m/b)v_0 + C_2$, so $C_2 = x_0 + (m/b)v_0$. Plugging these into $x(t)$ gives

$$x(t) = \frac{m}{b} v_0 \left(1 - e^{-(b/m)t}\right) + x_0$$

Final Answer: $x(t) = (m/b)v_0(1 - e^{-(b/m)t}) + x_0$

(h) Check that each term in your equation has correct units.
Solution: The ratio m/b has units of s. That means the exponent is unitless, and that $(m/b)v_0$ has units of m, which gives every term units of m.

(i) Take $m = 2000$, $b = 1000$, $x_0 = 0$, and $v_0 = 20$ in SI units. Draw quick sketches of $x(t)$ and $v(t)$.
Solution:

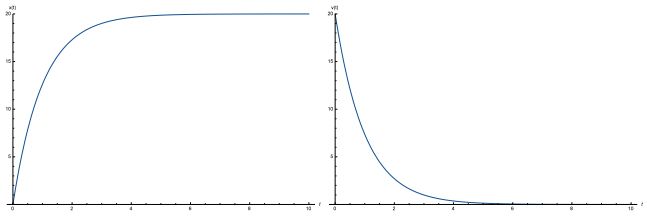


(j) Describe in words the motion your sketch shows. Does it make sense with the physical situation?

Solution: The car starts with an initial forward velocity, but gradually slows. It never completely stops, but asymptotically approaches a final position. It makes sense that it would slow down since the only force on it is air resistance. It also makes sense that it would slow down more gradually over time since the force on it decreases as it slows.

(k) Sketch the solution again with $b = 2000$ (and all other numbers the same). How did that affect the motion?

Solution:



The car slows down and stops more rapidly when there is more air resistance.

1.62 For each SHO below find the period, amplitude, and phase, or say which ones cannot be determined from the given information.

Solution: In all cases the given equation is in the form $x''(t) = -\omega^2 x(t)$ so the solution is in the form $x(t) = C\cos(\omega t + \phi)$. The given constant ω is the angular frequency, so the period is $2\pi/\omega$. If no initial conditions are given then the amplitude and phase could be literally anything. If initial conditions are given, we can use them to solve for C (the amplitude) and ϕ (the phase).

(a) $x''(t) = -9x$

Final Answer: Period $2\pi/3$, amplitude and phase undetermined

(b) $x''(t) = -25x$, $x(0) = 3$, $x'(0) = 0$

Final Answer: Period $2\pi/5$, amplitude 3, phase 0

(c) $x''(t) = qx$ where q is a negative constant

Final Answer: Period $2\pi/\sqrt{-q}$, amplitude and phase undetermined

(d) $x''(t) = -e^{2z}x$, $x(0) = 0$, $x'(0) = 1$ where z is a constant

Final Answer: Period $2\pi/e^z$, amplitude e^{-z}, phase $-\pi/2$

1.63 The simple harmonic oscillator equation $x''(t) = -\omega^2 x(t)$ (where ω is a constant) has solution $x(t) = C\cos(\omega t + \phi)$. Solve for C (the amplitude of oscillation) and ϕ (the phase) in terms of the initial conditions $x(0) = x_0$ and $v(0) = v_0$ and the angular frequency ω.

Final Answer: $C = \sqrt{x_0^2 + v_0^2/\omega^2}$ and $\phi = \tan^{-1}\left[-v_0/(\omega x_0)\right]$.

1.64 For each SHO below find the period, amplitude, and phase, or say which ones cannot be determined from the given information.

(a) A 5 kg mass is on a spring with spring constant 10 N/m. The force exerted by a spring is $-kx$, where k is the spring constant and x is the displacement from equilibrium.
Solution: We begin by combining Newton's second law $F = ma$ with the ideal spring law $F = -kx$ to write $ma = -kx$.

$$5x''(t) = -10x(t)$$

Dividing both sides by 5 turns this into the SHO equation $x''(t) = -2x(t)$ with $\omega^2 = 2$, so the period is $\pi\sqrt{2}$ seconds. Without initial conditions we can say nothing about the amplitude or phrase.
Final Answer: Period $\pi\sqrt{2}$, amplitude and phase undetermined

(b) The system in Part a is pulled out by a distance of 0.2 m and released from rest.
Solution: Now we have $x(0) = 0.2$ and $x'(0) = 0$. Plugging them into the solution $x(t) = C\cos(\sqrt{2}\,t + \phi)$ we can solve to find $C = 0.2$ and $\phi = 0$.
Final Answer: Period $\pi\sqrt{2}$, amplitude 0.2, phase 0

(c) A 2 m pendulum. A simple pendulum approximately obeys the equation $\theta''(t) = -(g/L)\theta$ where $g = 9.8$ m/s^2 and L is the length of the pendulum.
Solution: In this case $\omega^2 = g/L = 4.9$, so the period is $2\pi/\sqrt{4.9}$.
Final Answer: Period $2\pi/\sqrt{4.9}$, amplitude and phase undetermined

(d) A 2 m pendulum starts at $\theta = 0$ with initial angular velocity $\theta'(t) = 3$ s^{-1}.
Final Answer: Period $2\pi/\sqrt{4.9}$, amplitude $3/\sqrt{4.9}$, phase $-\pi/2$

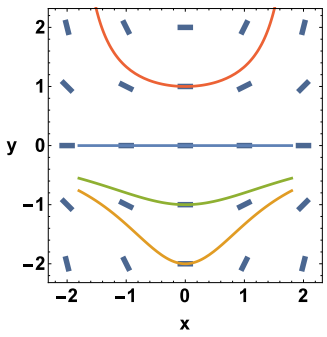
1.65 Walk-Through: Slope Fields. The function $y(x)$ evolves according to the differential equation $dy/dx = xy^2/2$.

(a) At the point $(-2, -2)$, the differential equation predicts a slope of -4. Draw a small line with a slope of -4 at this point. Your slope will not be exact, and that's OK! You can't visually distinguish a slope of -4 from -5, but the slope should clearly be much steeper than -1.
Solution: See below.

(b) Draw similar lines with the appropriate slopes at all integer points on the domain $x \in [-2, 2], y \in [-2, 2]$: a total of 25 points in all.
Solution: See below.

(c) Starting at the origin, draw a curve that extends both to the right and the left, using your slope lines as a guide.
Solution: See below.

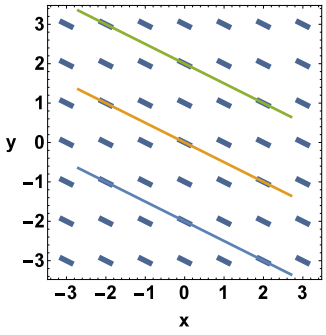
(d) Draw three more curves, going through the points $(0, -2)$, $(0, -1)$, and $(0, 1)$. In each case, the detailed points you hit are not important, but the overall shape is: you should be able to see at a glance where the curve is increasing and decreasing, roughly where it reaches a local minimum, and so on.
Solution:



(e) Based on your slope field, what is $\lim_{x\to\infty} y(x)$ if $y(0) = -1$?
Final Answer: 0

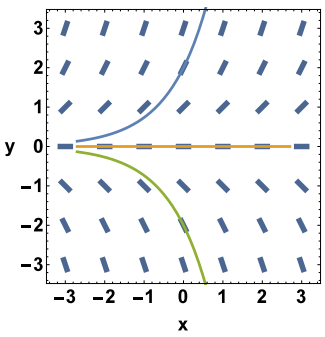
1.66 $dy/dx = -1/2$

Solution:



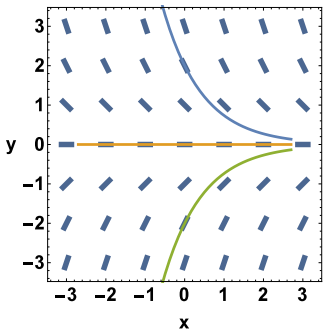
1.67 $dy/dx = y$

Solution:



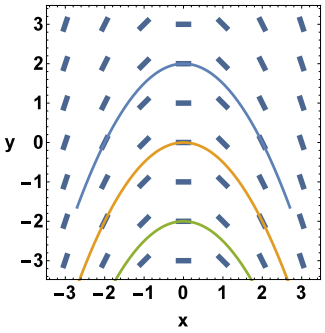
1.68 $dy/dx = -y$

Solution:



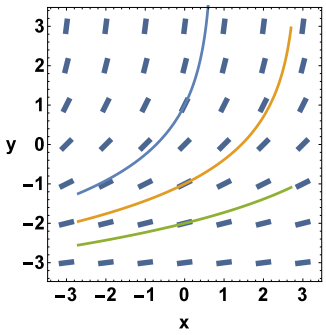
1.69 $dy/dx = -x$

Solution:



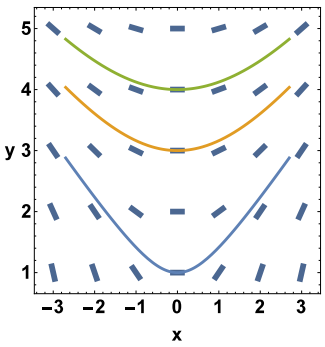
1.70 $dy/dx = 2^y$

Solution:



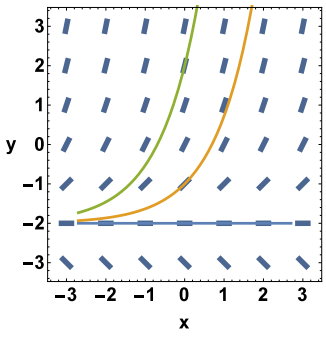
1.71 $dy/dx = x/y$ *(For this problem use the range* $1 \leq y \leq 5$*.)*

Solution:



1.72 $dy/dx = y + 2$

Solution:



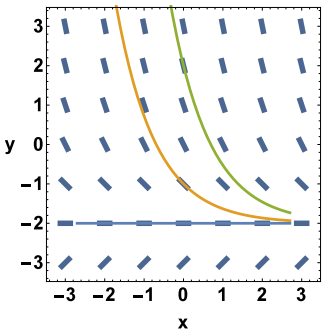
(c) $y = -2$ unstable

(e) If y starts at -2 it will stay at -2 forever. If it starts above or below -2 it will move away from -2 faster and faster over time.

Final Answer: (c) $y = -2$ unstable

1.73 $dy/dx = -(y+2)$

Solution:



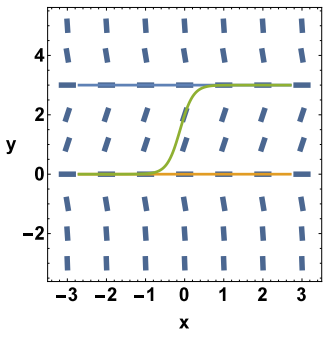
(c) $y = -2$ stable

(e) If y starts at -2 it will stay at -2 forever. If it starts above or below -2 it will move asymptotically towards -2.

Final Answer: (c) $y = -2$ stable

1.74 $dy/dx = 2y(3 - y)$

Solution:



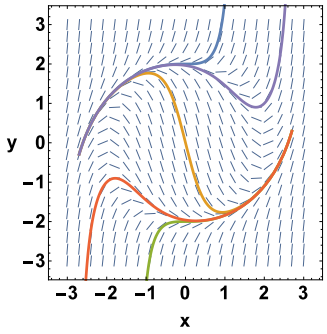
(c) $y = 0$ unstable, $y = 3$ stable

(e) If y starts at 0 it will stay there forever, and if it starts at 3 it will stay at 3 forever. If it starts above 0 at any value other than 3 it will asymptotically approach 3. If it starts below 0 it will get more negative faster and faster over time.

Final Answer: (c) $y = 0$ unstable, $y = 3$ stable

1.75 $dy/dx = x^2 + y^2 - 4$

Solution: Because the behavior of this system is fairly complicated it needs a lot of points in the slope field to see it clearly. We've also included five curves.



(a) slope is 0 at all points along the circle 2 units radially away from the origin

(c) There are no equilibrium solutions; there is no value of y where the slope always stays horizontal.

(e) If the system starts anywhere outside of the circle $x^2 + y^2 = 4$ then y will start moving upward. If it happens to pass through that circle then y will briefly decrease until it passes back out again, by $x = 2$ at the latest. After that it will rise forever.

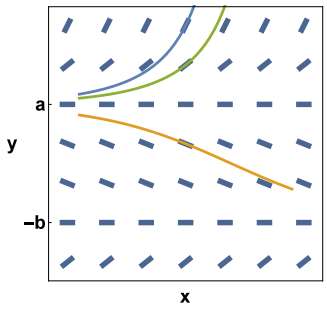
Final Answer: (c) no equilibrium solutions

1.77 Draw a slope field for the differential equation $dy/dx = k(y-a)(y+b)$ where k, a, and b are all positive constants. Since you don't have numerical values for these constants you can't know exactly what the actual slopes are, but you can still make a slope field that shows where the slopes are zero, positive or negative, and where they tend to be large or small. Label the equilibrium points on the y-axis.

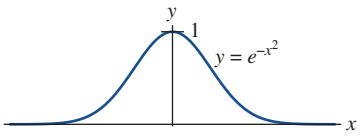
Solution: $k(y-a)(y+b)$ reaches zero at $y = -b$ and $y = a$, so those are the equilibrium values. Looking around those values…

- For $y < -b$ the slope is positive.
- For $-b < y < a$ the slope is negative.
- For $y > a$ the slope is positive.

We can therefore see that the equilibrium at $y = -b$ is stable. (If y is a bit less than $-b$ it will move up; if it is more, it will move down.) The equilibrium at $y = a$ is unstable. (If y is less than a it will move down, and if it is more it will move up.)

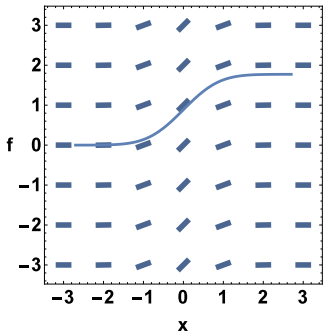


1.78 If you're not familiar with the wonderful function e^{-x^2}, there are two things you need to know for this problem. First, the equation $f'(x) = e^{-x^2}$ has no simple analytical solution. Second, the graph of $y = e^{-x^2}$ is shown below; it is always positive, reaches an absolute maximum at $(0, 1)$, and approaches 0 as $x \to \pm\infty$.



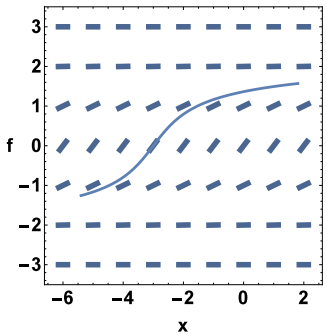
(a) Based on this graph, draw a slope field for the equation $f'(x) = e^{-x^2}$.
Solution: See below.

(b) Let $f(x)$ be the function that follows the differential equation $f'(x) = e^{-x^2}$ and contains the point $(-3, 0)$. Use your slope field to sketch a graph of $y = f(x)$.
Solution:



(c) Use your sketch to answer the question: What is the x-value of the point of inflection of $f(x)$?
Final Answer: $x = 0$

(d) Repeat Parts a–b for the differential equation $f'(x) = e^{-f^2}$. Then answer the question: what is the y-value of the point of inflection?
Solution:



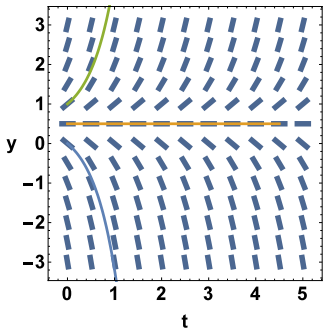
Final Answer: $y = 0$

1.79 The equations $dy/dt = 2y - 1$ and $dy/dt = 1 - 2y$ look a great deal alike, but lead to very different types of behavior. In this problem, you will draw slope fields and analyze the behavior of both equations; this will require two separate graphs.

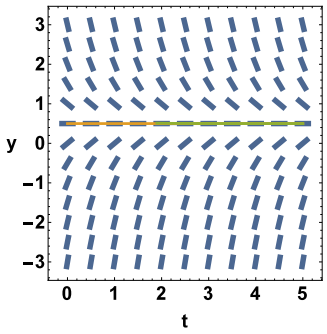
Note: because dy/dt has no t-dependence, you can ignore negative t-values without any loss of generality. So feel free to draw your slope fields only for positive t.

(a) For both of these graphs, $dy/dt = 0$ when $y = 1/2$. So begin both slope fields with horizontal lines at all points where $y = 1/2$.
Solution: See below.

(b) For $dy/dt = 2y - 1$, compute the slope when $y = 1$, and then draw in slope lines at all points where $y = 1$. Then repeat for $y = 2$, $y = 3$, $y = 0$, $y = -1$, and $y = -2$.
Solution:



(c) Repeat Part (b) for $dy/dt = 1 - 2y$.
Solution:



(d) Draw curves representing functions that begin at $(0, 0)$, $(0, 1/2)$, and $(0, 1)$ for both equations (six curves in all).
Solution: See above.

(e) For the equation $dy/dt = 2y - 1$, the function $y = 1/2$ represents an equilibrium solution: if y is ever 1/2, it will stay 1/2 forever. If y is "bumped" slightly above 1/2, how will it evolve over time? If y is "bumped" slightly below 1/2, how will it evolve over time? Does $y = 1/2$ represent a stable or unstable equilibrium?
Solution: If y is above $1/2$ it will move up and if it's below $1/2$ it will move down, so $y = 1/2$ is an unstable equilibrium.
Final Answer: Unstable

(f) Repeat Part e for the equation $dy/dt = 1 - 2y$.
Solution: If y is above $1/2$ it will move down towards $1/2$ and if it's below $1/2$ it will move up towards it, so $y = 1/2$ is a stable equilibrium.
Final Answer: Stable

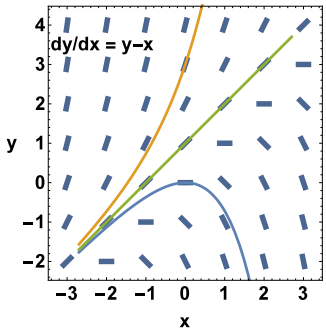
1.80 The equations $dy/dx = y - x$ and $dy/dx = x - y + 2$ both have the same linear solution $y = x + 1$, but behave quite differently otherwise. In this problem, you will draw slope fields and analyze the behavior of both equations; this will require two separate graphs.

(a) Show that $y = x + 1$ is a valid solution to both differential equations. (You will do this, not with a graph, but by plugging in.)
Solution: Plugging this solution into the first DE, you get $dy/dx = 1$ on the left side and $(x + 1) - x = 1$ on the right side. Plugging this solution into the second DE, you still get $dy/dx = 1$ on the left, and $x - (x + 1) + 2 = 1$ on the right.

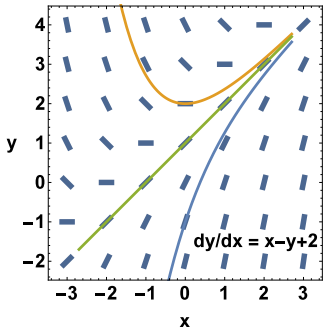
(b) If $y = x + 1$ is a solution to a differential equation, then dy/dx must equal 1 everywhere along that line. So begin both slope fields by drawing lines with a slope of 1 at the points $(-3, -2)$, $(-2, -1)$, and so on through $(3, 4)$.
Solution: See below.

(c) In the equation $dy/dx = y - x$, what happens to dy/dx if you increase y by one unit while leaving x unchanged? Based on your answer, draw in the slope lines at all points one unit higher than the points you drew in Part b.
Solution: See below for the graph.
Final Answer: dy/dx increases by 1

(d) Based on the same logic you used in Part c, draw in the slope lines one unit higher. Then move down to one unit, and then two units, below the line $y = x + 1$.
Solution:



(e) Repeat Parts c and d on your other graph for $dy/dx = x - y + 2$.
Solution:



(f) For the equation $dy/dx = y - x$, if the function starts on the line $y = x + 1$, it will stay on that line forever. How will a function evolve if it starts just above that line? Just below that line?
Solution: If it starts just above the line, the function will swerve away from the line in the upward direction and keep going up into the positive y-values. If it starts just below

the line, the function will swerve away from the line in the downward direction and keep going down into the negative y-values.

(g) For the equation $dy/dx = x - y + 2$, if the function starts on the line $y = x + 1$, it will stay on that line forever. How will a function evolve if it starts just above that line? Just below that line?

Solution: The function will asymptotically approach the line $y = x + 1$ whether it starts above or below that line.

1.81 In the Explanation (Section 1.4.2), we used a slope field to determine that the equation $dy/dx = y - 2x$ has only one linear solution, which is $y = 2x + 2$. In this problem, you will prove the same result analytically.

(a) If there is a linear solution, it can be written in the form $y = mx + b$. So plug $y = mx + b$ into the differential equation $dy/dx = y - 2x$.
Final Answer: $m = mx + b - 2x$

(b) The resulting equation sets two linear functions of x equal to each other. Put each function—the one on the left side of the equal sign, and the one on the right side—into the standard form of a line.
Final Answer: $2x + m = mx + b$

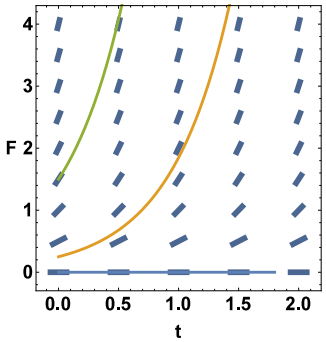
(c) If the left side of the equation is the same function as the right side, then their slopes (coefficients of x) must be equal, and their y-intercepts (constant parts) must also be equal. Use these two facts to solve for m and b.
Final Answer: $m = 2$ and $b = m = 2$

(Incidentally, an alternative approach is to remember that a line is the only function with zero concavity. So starting with $dy/dx = y - 2x$ you can set $d^2y/dx^2 = 0$ and reach the same conclusion.)

1.82 The population of the planet Foom is described by a function $F(t)$. Every Foomian has, on average, 2 babies per year.

(a) Assuming no Foomians ever die, write a differential equation for $F(t)$.
Final Answer: $dF/dt = 2F$

(b) Sketch a slope field for the equation you wrote down in Part (a). Since the number of Foomians can never be negative you only need to include values $F \geq 0$. Identify any equilibrium points in this range.
Solution:



Final Answer: $F = 0$ is an equilibrium point.

(c) Using this slope field, describe all the possible long-term behaviors of $F(t)$.
Solution: If the population starts out at $F = 0$, then it will always stay at $F = 0$ (you can't breed any Foomians if you don't have any to start with). If the population starts out at any value greater than 0, then $F(t)$ quickly blows up toward positive infinity.

1.83 A vicious rumor is spreading through a campus with a total population P. The number of students who have heard the rumor obeys the "logistic" differential equation $dS/dt = kS(P - S)$ where k is a positive constant.

(a) Draw a slope field for this equation. Because S cannot be smaller than 0 or larger than P you only need to include values of S in that range. Since you don't have numerical values for P and k you can't know exactly what the actual slopes are, but you can still make a slope field that shows where the slopes are zero, positive or negative, and where they tend to be large or small.

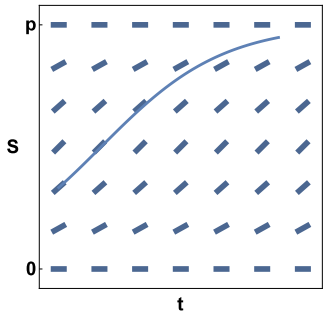
Solution: $kS(P - S)$ reaches zero at $S = 0$ and $S = P$, so those are the equilibrium values. Values of S that are below zero or above P are physically meaningless here. For all values $0 < S < P$ the slope is positive. But it is a small positive number (close to zero) when S is close to zero or close to P, and bigger in between. (See below for the plot.)

(b) For what values of S will $S(t)$ remain constant? (You can figure this out either from your slope field or directly from the differential equation.)

Final Answer: $S = 0$ and $S = P$

(c) Choose a value of $S(0)$ on your plot. (You don't need to pick a number; just mark a point on your plot.) Sketch the solution $S(t)$ with that initial condition on your slope field. Do *not* use a solution where $S(t)$ is constant.

Solution:



(d) Based on your slope field, what is $\lim_{t\to\infty} S(t)$?

Final Answer: P

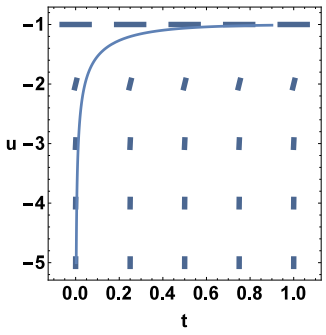
1.84 An object at $-200°C$ is placed in a $-1°C$ room. The object begins to warm, its temperature u following the differential equation $du/dt = u^4 - 1$.

(a) Explain why this equation, in this form, doesn't make sense for $u > 1$.

Solution: For $u > 1$, you predict that the object will cool down so that it would match the room temperature. However, the differential equation predicts that the object will warm up, which doesn't make sense.

(b) Using a slope field, sketch the temperature $u(t)$. Be sure to show the correct concavity at all times and the correct limit $\lim_{t\to\infty} u(t)$.

Solution:



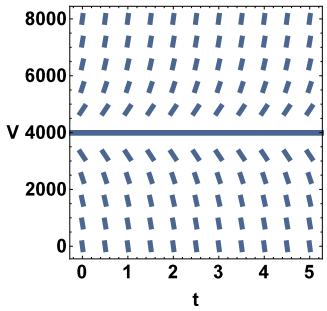
1.85 Venusians, the plucky inhabitants of the planet Venus, reproduce by a process called "quatosis": every year, every Venusian splits into four little Venusians. Left unchecked, the population of Venusians would soon overwhelm their planet. However, Martians kidnap 12,000 Venusians from the planet every year.

(a) Write the differential equation that governs the number of Venusians $V(t)$. Your differential equation should take into account both the gain due to quatosis, and the loss due to kidnapping.

Final Answer: $dV/dt = 3V - 12000$

(b) Draw a slope field for that differential equation. You can safely ignore negative t and V-values, which confines your slope field to one quadrant. Beyond that, part of your job is to choose enough V-values to see the behavior of this differential equation.

Solution:



(c) The equation has one equilibrium state; what is it?

Final Answer: $V = 4000$

(d) Is the equilibrium state stable or unstable? How can you tell?

Solution: For $V > 4000$ the differential equation predicts $dV/dt > 0$. So if the population is higher than 4000 it will grow. The higher the population gets, the faster it will grow.

For $V < 4000$ the differential equation predicts $dV/dt < 0$. So if the population is lower than 4000 it will shrink. The lower the population gets, the faster it will shrink.

So this is an unstable equilibrium. If it is precisely 4000 it can stay that way, but if it deviates from that level, the deviation will tend to increase over time.

Final Answer: unstable

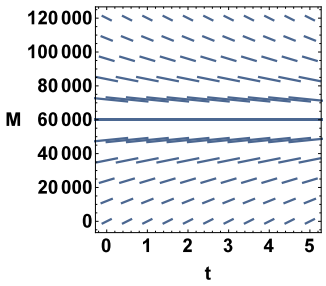
(e) Explain in words to a non-mathematician (from Earth) what will happen if the Venusian population starts higher than the equilibrium value, and why it will happen.

Solution: If the Venusian population starts higher than the equilibrium value, then the Venusian population will grow because more Venusians will be born each year than are kidnapped by the Martians. As the Venusian population grows, the birth rate increases while the rate at which Martians kidnap Venusians remains the same. Therefore, the Venusian population continues to grow at higher and higher rates.

1.86 The ancient Martian race is dying. The only way they keep their ancient culture going is by kidnapping; every year, they steal 12,000 Venusians. Through a high-tech process known as "Marsification" they transform their victims into Martians. Unfortunately, the process is unstable; the Martians cannot reproduce, and every year, one out of every five living Martians dies.

(a) Write the differential equation that governs the number of Martians $M(t)$. Your differential equation should take into account both the loss due to death, and the gain due to Marsification.
Final Answer: $dM/dt = 12000 - (1/5)M$

(b) Draw a slope field for that differential equation. You can safely ignore negative t and M-values, which confines your slope field to one quadrant. Beyond that, part of your job is to choose enough M-values to see the behavior of this differential equation.
Solution:



(c) The equation has one equilibrium state; what is it?
Final Answer: $M = 60000$

(d) Is the equilibrium state stable, or unstable? How can you tell?
Solution: For $M > 60,000$ the differential equation predicts $dM/dt < 0$. So if the population is higher than 60,000 it will shrink. As it shrinks down toward 60,000 the rate of shrinking will slow.
For $M < 60,000$ the differential equation predicts $dM/dt > 0$. So if the population is lower than 60,000 it will grow. As it grows toward 60,000 the growth rate will slow.
So this is a stable equilibrium. No matter where the population starts, it will trend toward 60,000 over time.
Final Answer: stable

(e) Explain in words to a non-mathematician (from Earth) why your answer to Part d makes sense. In other words, how could you have looked at our description of Mars and predicted without doing any math whether the equilibrium would be stable or unstable?
Solution: The Martian death rate is proportional to its own population. If the population grows too much, then the death rate will be high and the population will decrease until it is stable again. And if the population becomes too small, then the death rate will become lower than the rate of Marsification, and the population will be given a chance to grow again until it starts to approach the equilibrium value.

1.87 Newton's law of heating and cooling says that the rate of change of the temperature Q of an object is proportional to the difference between the temperature of the object and the temperature Q_R of the room it's in.

(a) Write this law as a differential equation for Q. Your equation will have a constant of proportionality in it, which you can call k. Assume k is positive and make sure the signs in your equation make sense physically. (Does it behave correctly when $Q > Q_R$ and when $Q < Q_R$?)

Solution: The "difference between Q and Q_R" could be written as either $Q - Q_R$ or $Q_R - Q$. In this case we want the latter form.

$$\frac{dQ}{dt} = k(Q_R - Q)$$

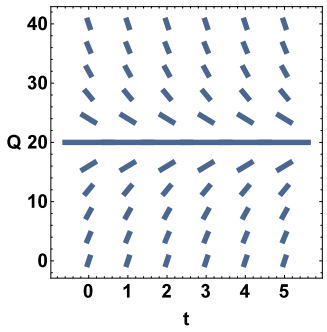
Why? If $Q < Q_R$ (the object is cooler than the room) then this equation correctly predicts that dQ/dt will be positive (the object will heat up). Conversely, if $Q > Q_R r$ (the object is warmer than the room) then this equation correctly predicts that dQ/dt will be negative (the object will cool down).

The equation $dQ/dt = k(Q - Q_R)$ can also be used for Newton's law, but it gives you a negative value for k, which we would rather avoid.

Final Answer: $dQ/dt = k(Q_r - Q)$

(b) Let $k = 1\ \text{s}^{-1}$ and $Q_R = 20°\text{C}$. Draw a slope field for Q. Choose a range of values for Q and t that clearly shows the possible behaviors of the system.

Solution:



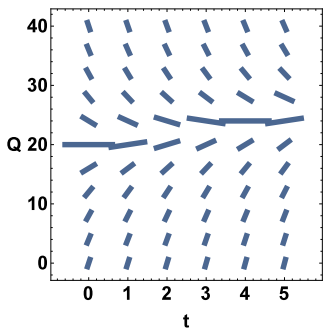
Now suppose the same object is in the same room, but now the room is steadily heating up: $Q_R = 20 + t$ with temperature in degrees Celsius and time in seconds.

(c) Rewrite the differential equation. (Your answer should have t in it, not Q_R.)

Final Answer: $dQ/dt = t + 20 - Q$

(d) Draw the slope field for this new situation.

Solution:



(e) Your slope field should show that there is an "attractor solution," a linear $Q = at + b$ that the temperature approaches no matter how it starts out. Using your slope field as a guide and some algebraic trial and error, find that linear solution and verify that it works in the differential equation.

Solution: Looking at the slope field we find that along the line $Q = t + 20$ the slope is always zero. Immediately below that line the slope is always 1. So the line $Q = t + 19$ is the solution we are looking for; if the object is on that line then it will rise with a slope of 1, which keeps it on that line.

To verify this solution note that $Q = t + 19$ leads to $dQ/dt = 1$ and also to $t + 20 - Q = 1$, so both sides of the differential equation give the same result.

Final Answer: $Q = t + 19$

(f) Describe what the temperature of the object does over time if $Q(0) = 20°C$.

Solution: At that point the slope is horizontal. Why? The temperature of the object is not changing because it is the same temperature as the room.

As we move to the right on the slope field, the slope becomes positive. Why? The object was not warming up but the room was, so the room is now slightly warmer than the object. This in turn causes the object to start to warm.

Over time the gap between the room temperature and the object temperature will approach 1. That is, the object will continue to rise in temperature but will lag the room temperature by 1° C.

1.88 Walk-Through: Separation of Variables. In this problem, you are going to solve the equation $dx/dt = te^x$.

(a) Multiply both sides of the equation by $e^{-x}dt$. This "separates the variables": the left side has only x-dependence, and the right side t.
Final Answer: $e^{-x}dx = t\,dt$

(b) Write an integral sign on both sides of the equation.
Final Answer: $\int e^{-x}dx = \int t\,dt$

(c) Integrate both sides. Include a $+C$ in your answer on the right, but not on the left.
Final Answer: $-e^{-x} = (1/2)t^2 + C$

(d) Solve for x.
Final Answer: $x = -\ln(C - t^2/2)$

(e) Demonstrate that you have found a valid solution to the differential equation.
Solution:

$$\begin{aligned} x &= -\ln\left(C - \frac{1}{2}t^2\right) \\ &\rightarrow \frac{dx}{dt} = -\frac{1}{C - t^2/2} \times (-t) \\ &= \frac{t}{C - t^2/2} \\ &= -\ln\left(C - \frac{1}{2}t^2\right) \\ &\rightarrow te^x = te^{-\ln(C - t^2/2)} = \frac{t}{e^{\ln(C - t^2/2)}} \\ &= \frac{t}{C - t^2/2} \end{aligned}$$

They come out the same, so it satisfies the differential equation.

(f) Find the specific function that satisfies the differential equation $dx/dt = te^x$ and the initial condition $x(0) = 1$.
Final Answer: $x = -\ln(1/e - t^2/2)$

1.89 $dy/dx = 2x$

Solution: Multiply both sides by dx to separate variables. Then integrate both sides, including an arbitrary constant on one side only.

$$\int dy = \int 2x\, dx$$
$$y = x^2 + C$$

To demonstrate that it satisfies the differential equation, take the derivative of the solution. You get $dy/dx = 2x$, which is the same as the differential equation.

Final Answer: $y = x^2 + C$

1.90 $dy/dx = 2y$

Solution: Multiply both sides by $(1/y)dx$ to separate variables. Integrate both sides and then solve for y.

$$\int \frac{1}{y} dy = \int 2\, dx$$
$$\ln y = 2x + C$$
$$y = e^{2x+C} = e^{2x} e^{C}$$

Since e^C is just another arbitrary constant, you can call it A.

$$y = Ae^{2x}$$

When you take the derivative of this solution, you get $dy/dx = 2Ae^{2x} = 2y$, which is the same as the original differential equation. Therefore, this solution satisfies the differential equation.

Final Answer: $y = Ae^{2x}$

1.91 $dy/dx = xy$

Solution: Multiply both sides by $(1/y)dx$. Integrate both sides and then solve for y.

$$\int \frac{1}{y} dy = \int x\, dx$$
$$\ln y = \frac{1}{2}x^2 + C$$
$$y = e^{x^2/2+C} = e^{x^2/2}e^C$$
$$y = Ae^{x^2/2}$$

When you take the derivative of this solution, you get $dy/dx = Ae^{x^2/2} \times x = yx$, which is the same as the original differential equation. So the solution satisfies the differential equation.

Final Answer: $y = Ae^{x^2/2}$

1.92 $dz/dt = 3z + 6$

Solution: You can begin by dividing both sides by $3z + 6$, but then you have to be careful when integrating. You're less likely to make mistakes if you leave the 3 on the right side, as follows.

$$\begin{aligned} \frac{dz}{dt} &= 3(z+2) \\ \int \frac{1}{z+2} dz &= \int 3\, dt \\ \ln(z+2) &= 3t + C \\ z + 2 &= e^{3t+C} = e^{3t} e^{C} \end{aligned}$$

Since e^C is just another arbitrary constant, you can call it A.

$$z = Ae^{3t} - 2$$

To check that it satisfies the differential equation, plug this solution into both sides of the differential equation. Both the LHS and the RHS give you $3Ae^{3t}$, so this solution satisfies the differential equation.

Final Answer: $z = Ae^{3t} - 2$

1.93 $dz/dt = 8 - 5z$

Solution: You can begin by dividing both sides by $8 - 5z$, but then you have to be careful when integrating. You're less likely to make mistakes if you leave the -5 on the right side, as follows.

$$\begin{aligned} \frac{dz}{dt} &= -5\left(z - \frac{8}{5}\right) \\ \int \frac{1}{z - 8/5} dz &= \int -5dt \\ \ln\left(z - \frac{8}{5}\right) &= -5t + C \\ z - \frac{8}{5} &= e^{-5t+C} = e^{-5t}e^{C} \end{aligned}$$

Since e^C is just another arbitrary constant, you can call it A.

$$z = Ae^{-5t} + \frac{8}{5}$$

To check that it satisfies the differential equation, plug this solution into both sides of the differential equation. Both the LHS and the RHS give you $-5Ae^{-5t}$, so this solution satisfies the differential equation.

Final Answer: $z = Ae^{-5t} + 8/5$

1.94 $dx/dt = (t^2 + 1)/(2t)$

Solution:

$$\int dx = \int \frac{t^2 + 1}{2t} dt$$

$$x = \int \left(\frac{t}{2} + \frac{1}{2t}\right) dt$$

$$x = \int \frac{t}{2} dt + \int \frac{1}{2t} dt$$

$$x = \frac{1}{4}t^2 + \frac{1}{2} \ln t + C$$

Plug this solution into both sides of the differential equation. The derivative $dx/dt = (t/2) + 1/(2t)$, and this can be simplified to $(t^2 + 1)/2t$, so this solution satisfies the differential equation.

Final Answer: $x = (1/4)t^2 + (1/2) \ln t + C$

1.95 $dx/dt = (x^2 + 1)/(2x)$

Solution: Multiply both sides by $2x/\left(x^2+1\right)dt$ to separate variables. Then integrate both sides, including an arbitrary constant on one side only, and then solve for x.

$$\int \frac{2x}{x^2+1}dx = \int dt$$
$$\ln\left(x^2+1\right) = t + C$$
$$x^2 + 1 = e^{t+C} = e^t e^C$$
$$x = \pm\sqrt{e^t e^C - 1}$$

Since e^C is just another arbitrary constant, you can call it A.

$$x = \pm\sqrt{Ae^t - 1}$$

We will confirm the positive solution; the math looks very similar for the negative.

$$x = \sqrt{Ae^t - 1} \quad \rightarrow \quad \frac{dx}{dt} = \frac{Ae^t}{2\sqrt{Ae^t - 1}}$$
$$x = \sqrt{Ae^t - 1} \quad \rightarrow \quad \frac{x^2+1}{2x} = \frac{Ae^t}{2\sqrt{Ae^t - 1}}$$

Both sides come out the same, so the function solves the differential equation.

Final Answer: $x = \pm\sqrt{Ae^t - 1}$

1.96 $\dfrac{dA}{d\theta} = \dfrac{\tan\theta}{A^2\sqrt{A^3+\pi}}$

Solution:

$$\int A^2\sqrt{A^3+\pi}\,dA = \int \tan\theta\,d\theta$$

You can integrate the left side by first re-writing the integral as $\int A^2\left(A^3+\pi\right)^{1/2}dA$ and then use $u = A^3+\pi$ to end up with $(2/9)\left(A^3+\pi\right)^{(3/2)}$.

You can integrate the right side by first re-writing the integral as $\int(\sin\theta)/(\cos\theta)d\theta$ and then use $u = \cos\theta$ to arrive at the result $-\ln(\cos\theta)+C$.

Going back to your first equation, you can now evaluate both integrals to get:

$$\frac{2}{9}\left(A^3+\pi\right)^{3/2} = -\ln(\cos\theta)+C$$

Now you can solve for A.

$$A^3 = \left(-\frac{9}{2}\ln(\cos\theta)+C\right)^{2/3} - \pi$$

To confirm this answer, take the derivative of both sides with respect to θ.

$$3A^2\frac{dA}{d\theta} = \frac{2}{3}\left(-\frac{9}{2}\ln(\cos\theta)+C\right)^{-1/3}\left(\frac{9}{2}\tan\theta\right)$$
$$A^2\frac{dA}{d\theta} = \tan\theta\left(-\frac{9}{2}\ln(\cos\theta)+C\right)^{-1/3}$$

Does that match the original differential equation? It does! Note that for the A^3 we are verifying,

$$\frac{1}{\sqrt{A^3+\pi}} = \left(-\frac{9}{2}\ln(\cos\theta)+C\right)^{-1/3}$$

Final Answer: $A^3 = (-(9/2)\ln(\cos\theta)+C)^{2/3} - \pi$

1.97 $\dfrac{dr}{d\theta} = \dfrac{\sqrt{r}\,\cos(2\theta)}{e^{\sqrt{r}}}$

Solution:

$$\int \frac{e^{\sqrt{r}}}{\sqrt{r}} dr = \int \cos(2\theta)\, d\theta$$

$$2e^{\sqrt{r}} = \frac{1}{2}\sin(2\theta) + C$$

$$r = \left[\ln\left(\frac{1}{4}\sin(2\theta) + C\right)\right]^2$$

To check, plug this solution into both sides of the differential equation. You get

$$\frac{dr}{d\theta} = \frac{\ln\left(\frac{1}{4}\sin(2\theta) + C\right) \times \cos(2\theta)}{\frac{1}{4}\sin(2\theta) + C}$$

on the left side and the same thing when you plug in the expression for r on the right side. Therefore, it satisfies the differential equation.

Final Answer: $r = \left[\ln\left(\frac{1}{4}\sin(2\theta) + C\right)\right]^2$

1.98 $dy/dx = x\sqrt{9-y^2}$

Solution:

$$\int \frac{1}{\sqrt{9-y^2}}dy = \int x\,dx$$

To solve the left hand integral, let $y = 3\sin\theta$, where $-(\pi/2) \le \theta \le (\pi/2)$. After making this substitution and evaluating the integral, you will get $\sin^{-1}(y/3)$.

$$\sin^{-1}\left(\frac{y}{3}\right) = \frac{1}{2}x^2 + C$$
$$y = 3\sin\left(\frac{1}{2}x^2 + C\right)$$

To confirm this we plug it into both sides of the equation.

$$\frac{dy}{dx} = 3\cos\left(\frac{1}{2}x^2 + C\right) \times x$$
$$x\sqrt{9-x^2} = x \times \sqrt{9 - 9\sin^2\left(\frac{1}{2}x^2 + C\right)}$$

Factor out the 9 (it becomes a 3 outside the square root) and you are left with $\sqrt{1-\sin^2}$ which becomes cos and then the two look the same.

Final Answer: $y = 3\sin\left(x^2/2 + C\right)$

1.99 $ds/dt = (s^3 + 3)/s^2 \times \ln t$

Solution:

$$\int \frac{s^2}{s^3+3} ds = \int \ln t \, dt$$

To solve the left hand integral, substitute $u = s^3 + 3$ and you get $(1/3)\ln(s^3 + 3)$.

To solve the right hand integral, let $u = \ln t$ and $dv = dt$ and perform integration by parts. You get $t \ln t - t + C$.

$$\frac{1}{3}\ln(s^3+3) = t\ln t - t + C$$

$$s^3 + 3 = \frac{At^{3t}}{e^{3t}}$$

$$s = \left(\frac{At^{3t}}{e^{3t}} - 3\right)^{1/3}$$

To verify this solution, plug it into both sides of the differential equation. Note that one step requires taking the derivative of t^{3t}; one way to do this is to write $t^{3t} = e^{3t \ln t}$ so its derivative is $t^{3t} \times (3 + 3\ln t)$.

$$\frac{ds}{dt} = \frac{1}{3}\left(\frac{At^{3t}}{e^{3t}} - 3\right)^{-2/3}\left(\frac{Ae^{3t}t^{3t}(3+3\ln t) - 3At^{3t}e^{3t}}{e^{6t}}\right)$$

$$= A\left(\frac{At^{3t}}{e^{3t}} - 3\right)^{-2/3}\left(\frac{t^{3t}\ln t}{e^{3t}}\right)$$

$$\frac{s^3+3}{s^2}\ln t = \frac{At^{3t}}{e^{3t}}\left(\frac{At^{3t}}{e^{3t}} - 3\right)^{-2/3}\ln t$$

They are the same, so it works.

Final Answer: $s = (At^{3t}/e^{3t} - 3)^{1/3}$

1.100 $dx/dt = \cos(\omega t)(x^2 + 25)$ (ω is a constant)

Solution:

$$\int \frac{1}{(x^2 + 25)} dx = \int \cos(\omega t)\, dt$$

The integral on the left can be approached with the substitution $x = 5\tan\theta$. We will skip the next few steps and jump to the solution.

$$\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) = \frac{1}{\omega}\sin(\omega t) + C$$

$$x = 5\tan\left(\frac{5}{\omega}\sin(\omega t) + C\right)$$

To verify this solution, plug it into both sides of the differential equation.

$$\frac{dx}{dt} = 5\sec^2\left(\frac{5}{\omega}\sin(\omega t) + C\right) \times 5\cos(\omega t)$$

$$\cos(\omega t)(x^2 + 25) = \cos(\omega t)\left[25\tan^2\left(\frac{5}{\omega}\sin(\omega t) + C\right) + 25\right]$$

With the identity $\sec^2 y = \tan^2 y + 1$ we can quickly see that the two functions are the same, so the solution works.

Final Answer: $x = 5\tan\left[(5/\omega)\sin(\omega t) + C\right]$

1.101 $dy/d\theta = y^2 \sin^3(\theta)$

Solution:

$$\int \frac{1}{y^2} dy = \int \sin^3\theta \, d\theta$$

To evaluate the right hand integral, you can use the following method:

$$\int \sin^3\theta \, d\theta = \int \sin^2\theta \sin\theta \, d\theta = \int \left(1 - \cos^2\theta\right) \sin\theta \, d\theta$$

Then substitute $u = \cos\theta$ and the rest is easy. Evaluating both integrals, you get:

$$-\frac{1}{y} = \frac{1}{3}\cos^3\theta - \cos\theta + C$$

$$y = \frac{3}{3\cos\theta - \cos^3\theta + C}$$

To verify this solution, plug it into both sides of the differential equation.

$$\frac{dy}{d\theta} = -\frac{3(-3\sin\theta + 3\sin\theta\cos^2\theta)}{(3\cos\theta - \cos^3\theta + C)^2}$$

$$y^2\sin^3\theta = \frac{9\sin^3\theta}{(3\cos\theta - \cos^3\theta + C)^2}$$

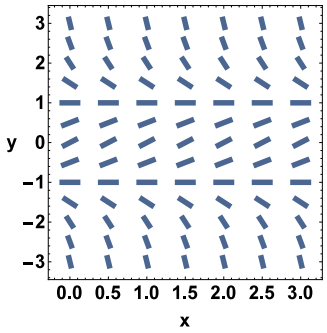
Factor out the common $3\sin\theta$ in the numerator and then turn $1 - \cos^2\theta$ into $\sin^2\theta$ to see that the two functions are the same.

Final Answer: $y = 3/(3\cos\theta - \cos^3\theta + C)$

1.102 $dy/dx = 1 - y^2$

(a) Create a slope field for all integer points $0 \le x \le 3$, $-3 \le y \le 3$.

Solution:



(b) Based on your slope field, describe the behavior of $y(x)$ if…

i. $y(0) > 1$

Solution: The function will decrease and approach $y = 1$.

ii. $y(0) = 1$

Solution: The function will remain at $y = 1$, so the graph will be a horizontal line.

iii. $-1 < y(0) < 1$

Solution: The function will increase, with the greatest slope at $y = 0$, and then approach $y = 1$.

iv. $y(0) = -1$

Solution: The function will remain at $y = -1$, so the graph will be a horizontal line.

v. $y(0) < -1$

Solution: The function will decrease to negative y-values without bound.

(c) Solve the original equation to find a function $y(x)$. (The easiest way to do this by hand requires the technique of partial fractions for your integral. Alternatively you could set up the integrals by hand and then evaluate them with a computer.)

Solution:

$$\int \frac{1}{1-y^2} dy = \int dx$$

Rewrite this using partial fractions.

$$\frac{1}{1-y^2} = \frac{1}{(1+y)(1-y)} = \frac{A}{(1+y)} + \frac{B}{(1-y)}$$

You can solve for A and B to get $A = B = (1/2)$.

$$\int \frac{1}{2}\frac{1}{(1+y)}dy + \int \frac{1}{2}\frac{1}{(1-y)}dy = x + C$$
$$\frac{1}{2}\ln\frac{1+y}{1-y} = x + C$$
$$\frac{1+y}{1-y} = Ae^{2x}$$
$$1 + y = Ae^{2x} - Ae^{2x}y$$
$$Ae^{2x}y + y = Ae^{2x} - 1$$
$$y(Ae^{2x} + 1) = Ae^{2x} - 1$$
$$y = \frac{Ae^{2x} - 1}{Ae^{2x} + 1}$$

Final Answer: $y = (Ae^{2x} - 1)/(Ae^{2x} + 1)$

(d) Show that your resulting $y(x)$ function is a valid solution to the original differential equation.

Solution:

$$\frac{dy}{dx} = \frac{2Ae^{2x}(Ae^{2x} + 1) - 2Ae^{2x}(Ae^{2x} - 1)}{(Ae^{2x} + 1)^2}$$
$$= \frac{4Ae^{2x}}{(Ae^{2x} + 1)^2}$$
$$1 - y^2 = \frac{(Ae^{2x} + 1)^2}{(Ae^{2x} + 1)^2} - \frac{(Ae^{2x} - 1)^2}{(Ae^{2x} + 1)^2}$$
$$= \frac{4Ae^{2x}}{(Ae^{2x} + 1)^2}$$

(e) Find the particular solutions corresponding to the initial conditions $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$. This will involve either finding the right value of the arbitrary constant or allowing it to approach ∞.

Solution: For $y(0) = 0$, plug 0 for both x and y in the equation for $-1 < y < 1$, and you will get $A = 1$. So the particular solution is $y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$.

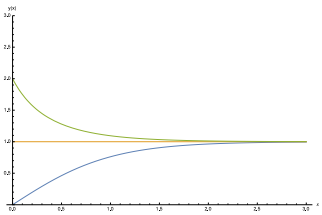
For $y(0) = 1$, the particular solution is $y = 1$. We reach this solution from the general solution by letting $A \to \infty$.

For $y(0) = 2$, plug 0 for x and 2 for y in the equation for $y > 1$, and you will get $A = 3$. So the particular solution is $y = \dfrac{3e^{2x} + 1}{3e^{2x} - 1}$.

Final Answer: For $y(0) = 0$, $y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$. For $y(0) = 1$, $y = 1$. For $y(0) = 2$, $y = \dfrac{3e^{2x} + 1}{3e^{2x} - 1}$.

(f) Plot the three particular solutions you just found. You should be able to see that they display the behaviors you described in Part (b).

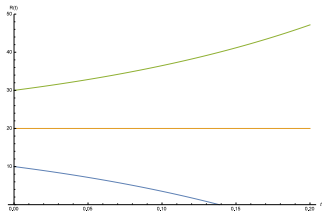
Solution:



1.103 In the Explanation (Section 1.5.2) we solved the equation $dR/dt = 5R - 100$ that describes a population of rabbits that is reproducing and being hunted. We found the solution $R = Ce^{5t} + 20$.

(a) Find the specific solutions for $R(0) = 10$, $R(0) = 20$, and $R(0) = 30$.
Final Answer: For $R(0) = 10$, $R = -10e^{5t} + 20$. For $R(0) = 20$, $R = 20$. For $R(0) = 30$, $R = 10e^{5t} + 20$.

(b) Sketch the three solutions you found together on one plot.
Solution:



(c) For each of these solutions, explain why the behavior you found (increasing, decreasing, or staying constant) makes sense. Don't answer in terms of equations and variables; explain it in terms of rabbits and hunters!
Solution: For $R(0) = 10$, there are too few rabbits present initially to reproduce a significant amount, so the birth rate is smaller than the death rate due to hunting. This causes the population to decrease. As the population decreases, there are less rabbits to mate with each other, so the birth rate decreases while the death rate stays the same. The population decreases faster and faster until all the rabbits die out.
For $R(0) = 20$, there are just enough rabbits present initially to mate with each other so that the birth rate is equal to the death rate. Therefore, the population stays the same.
For $R(0) = 30$, there are more than enough rabbits present initially to mate with each other, so the birth rate is greater than the death rate. This causes the population to increase. As the population increases, there are more and more rabbits to mate with each other, so the birth rate increases while the death rate stays the same. The population therefore grows exponentially.

(d) Identify the equilibrium solution for this equation. Is it stable or unstable?
Final Answer: $R = 20$; unstable

1.104 According to NASA, the Arctic sea ice extent is melting at a rate of 10% per decade. Suppose that at just the moment when the ice extent reaches 25 Million square kilometers of ice, a crack team of engineers begins replenishing the ice at a constant rate of 1 Million km^2 per decade.

Let P equal the size of the Arctic sea ice extent, measured in millions of square kilometers. Let t equal the time, measured in decades.

(a) Write a differential equation for $P(t)$.
Final Answer: $dP/dt = -.10P + 1$

(b) Find the equilibrium solution to your equation. Does it represent a stable or unstable equilibrium? How can you tell?
Solution: Setting dP/dt equal to 0 in the differential equation, you get $P = 10$. So, whenever the value of P reaches 10, it will stay 10 forever. The equilibrium solution is then $P = 10$. You notice that for all $P > 10$, dP/dt is negative, and for all $P < 10$, dP/dt is positive. This tells you that if you start above the $P = 10$ line, P will decrease until it starts to approach $P = 10$. If you start below the $P = 10$ line, your P will increase until it starts to approach $P = 10$. Therefore, you have a stable equilibrium.
Final Answer: $P = 10$; stable

(c) Solve the differential equation, with initial condition, to find the function $P(t)$.
Solution:

$$\int \frac{1}{-.10P + 1} dP = \int dt$$
$$-10 \ln(-.10P + 1) = t + C$$
$$P = Ae^{-.10t} + 10$$

You started measuring your time when the engineers began replenishing the ice extent, at which point $P = 25$. So you can use the initial condition $P(0) = 25$ to find the particular solution. You will get $P = 15e^{-.10t} + 10$.
Final Answer: $P = 15e^{-.10t} + 10$.

(d) Verify that your solution solves the differential equation.
Solution: $dP/dt = -1.5e^{-.10t}$, and plugging the equation for P into the right hand side of the differential equation you get the same thing.

1.105 James has just jumped out of an airplane. After he opens his parachute he experiences two forces: the constant force of gravity, and a wind drag that is proportional to his velocity. His height may therefore follow the equation:

$$\frac{d^2h}{dt^2} = -9.8 - 2\frac{dh}{dt} \tag{1.5.4}$$

As a second-order differential equation, this is not technically solvable by separation of variables. However, because the variable h appears *only* in its derivatives, we can turn this into a first-order equation, and solve that by separation.

(a) Letting $v = dh/dt$, rewrite Equation 1.5.4 as a first-order differential equation in v.
Final Answer: $dv/dt = -9.8 - 2v$

(b) Your equation in Part a suggests that there is one velocity for which $dv/dt = 0$. What is this velocity?
Final Answer: $v = -4.9$

(c) Solve your equation using separation of variables. Your solution should contain an arbitrary constant: call it C_1.
Final Answer: $v(t) = C_1 e^{-2t} - 4.9$

(d) Calculate $\lim_{t\to\infty} v(t)$ and use it to describe what is physically happening to James after he's been in the air for a long time.
Solution: $\lim_{t\to\infty} v(t) = -4.9$
The faster James goes, the more the air pushes back on him. When he reaches 4.9 m/s the force of air resistance exactly cancels the force of gravity. This is his "terminal velocity"; he will continue to fall at this rate.
Final Answer: -4.9

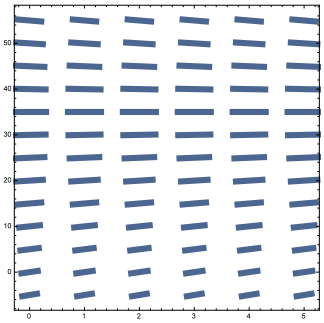
(e) Now that you have a velocity function $v(t)$, integrate it with respect to t to find a position function $h(t)$. This will introduce a second constant C_2.
Final Answer: $x(t) = -(C_1/2)e^{-2t} - 4.9t + C_2$

(f) Supposing James begins at a height of 3000 m with no initial velocity, what is his height 3 s later?
Solution: Applying the condition $v(0) = 0$ gives $C_1 = 4.9$. The remaining condition gives $3000 = -2.45 + C_2$, so $C_2 = 3002.45$. Finally, plugging in $t = 3$ gives $x(3) = -2.45e^{-6} - 4.9(3) + 3002.45$
Final Answer: $h(3) = 2988$ m

1.106 A frozen turkey placed in a 35°F refrigerator begins to thaw according to the equation $dQ/dt = (1/20)(35 - Q)$ where Q is the temperature of the turkey. Assume temperature is measured in degrees Fahrenheit and time is measured in hours.

(a) Draw a slope field for $0 \le t \le 5$ and $Q = -5, 15, 25, 35, 45, 55$.

Solution:



(b) Based on your slope field, what is the equilibrium solution to this equation? Is it a stable or unstable equilibrium?

Final Answer: 35°, stable

(c) Solve the differential equation by separation of variables, assuming the turkey started ($t = 0$) in a 0°F freezer.

Solution:

$$\int \frac{dQ}{Q-35} = -\int \frac{1}{20}dt$$
$$\ln(Q-35) = -\frac{t}{20} + C$$
$$Q = Ce^{-t/20} + 35$$
$$0 = C + 35$$
$$C = -35$$
$$Q = 35\left(1 - e^{-t/20}\right)$$

Final Answer: $Q(t) = 35(1 - e^{-t/20})$

(d) Based on your solution, what is $\lim_{t\to\infty} Q(t)$?

Final Answer: $\lim_{t\to\infty} Q(t) = 35$

(e) How long will it take the turkey to reach 32°F?

Final Answer: 49 hrs

1.107 The Expanding Universe

The universe is expanding according to "Hubble's Law":

$$\frac{da}{dt} = \sqrt{\frac{8\pi G}{3c^2}\rho}\ a \tag{1.5.5}$$

where a is the distance between two reference objects and ρ is the energy density of the universe. (Note: it will be easier to work with this equation if you collect the constants into one letter: say, $\beta = \sqrt{8\pi G/3c^2}$.)

(a) If we consider the universe to be primarily composed of *matter,* then $\rho = k/a^3$. (This makes sense if you think about it.) Solve the resulting differential equation.

Solution:

$$\frac{da}{dt} = \beta(ka^{-3})^{1/2}a$$
$$\frac{da}{dt} = \frac{\beta\sqrt{k}}{\sqrt{a}}$$
$$\int \sqrt{a}\,da = \int \beta\sqrt{k}\,dt$$
$$\frac{2}{3}a^{3/2} = (\beta\sqrt{k})t + C$$
$$a = \left(\frac{3}{2}\beta\sqrt{k}\ t + C\right)^{2/3}$$

Final Answer: $a = (3\beta\sqrt{k}\ t/2 + C)^{2/3}$

(b) If we consider the universe to be primarily composed of *radiation,* then $\rho = k/a^4$. Solve the resulting differential equation.

Solution:

$$\frac{da}{dt} = \beta(ka^{-4})^{1/2}a$$
$$\frac{da}{dt} = \frac{\beta\sqrt{k}}{a}$$
$$\int a\,da = \int \beta\sqrt{k}\,dt$$
$$\frac{1}{2}a^2 = (\beta\sqrt{k})t + C$$
$$a = \sqrt{2\beta\sqrt{k}\ t + C}$$

Final Answer: $a = \sqrt{2\beta\sqrt{k}\ t + C}$

(c) If we consider the universe to be primarily composed of *dark energy* (Einstein's "cosmological constant") then ρ is a constant. Solve the resulting differential equation.

Solution:

$$\frac{da}{dt} = \beta\sqrt{\rho}\,a$$
$$\int \frac{1}{a}da = \int \beta\sqrt{\rho}\,dt$$
$$\ln(a) = (\beta\sqrt{\rho})t + C$$
$$a = Ae^{\beta\sqrt{\rho}\ t}$$

Final Answer: $a = Ae^{\beta\sqrt{\rho}\ t}$

(d) Describe in words the difference between how a universe expands when it is filled with matter or radiation vs. how it expands when it is filled with dark energy.
Solution: A universe filled with matter or radiation expands as a power of time, and slows down over time. A universe filled with dark energy expands as an exponential of time and speeds up over time.

1.108 Walk-Through: Guess and Check. Consider the equation $\left(d^2y/dx^2\right) - 10\,(dy/dx) + 9y = 27$.

(a) Write the complementary homogeneous equation.
Final Answer: $d^2y/dx^2 - 10(dy/dx) + 9y = 0$

(b) Plug the "guess" $y = e^{kx}$ into the equation that you wrote in Part a. The result should be an algebraic equation.
Final Answer: $\left(k^2 - 10k + 9\right) e^{kx} = 0$

(c) Solve the resulting algebraic equation for k, and use the result to write two different solutions to the complementary equation.
Final Answer: $y_1 = e^x$, $y_2 = e^{9x}$

(d) By multiplying your answers to Part (c) by arbitrary constants, and adding the two resulting solutions together, write the general solution to the complementary equation.
Final Answer: $y_C = Ae^x + Be^{9x}$

(e) Find a simple solution—a constant—to the original inhomogeneous equation.
Final Answer: $y_P = 3$

(f) By adding your general solution from Part d to your particular solution from Part (d), write the general solution to the problem we started with.
Final Answer: $y(t) = Ae^x + Be^{9x} + 3$

(g) Demonstrate that your function from Part (f) solves the differential equation.
Solution: First take the derivatives: $dy/dt = Ae^x + 9Be^{9x}$ and $d^2y/dt^2 = Ae^x + 81Be^{9x}$, so

$$\begin{aligned}\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 9y &= Ae^x + 81Be^{9x} - 10\left(Ae^x + 9Be^{9x}\right) + 9\left(Ae^x + Be^{9x} + 3\right)\\ &= (A - 10A + 9A)\,e^x + (81B - 90B + 9B)\,e^{9x} + 27\\ &= 27\end{aligned}$$

1.109 Consider the problem $dy/dx = xy + y$. Suspecting an exponential solution, you decide to try $y = e^{kx}$.

(a) Plug the trial solution $y = e^{kx}$ into the differential equation $dy/dx = xy + y$.
Solution: $ke^{kx} = (x + 1)e^{kx}$

(b) The resulting equation can easily be solved to yield $k = x + 1$. Have you found a solution? Test $y = e^{x(x+1)}$ in the differential equation $dy/dx = xy + y$.
Solution: $(2x + 1)e^{x(x+1)} \stackrel{?}{=} (x + 1)e^{x(x+1)}$
This is not true unless $2x = x$. A solution that only works for $x = 0$ is no solution at all.

(c) It didn't work! Where did we go wrong?
Solution: When we said that we were assuming a solution of the form $y = e^{kx}$ we implicitly meant "where k is a constant." We relied on this assumption when we took the derivative of e^{kx} and got ke^{kx} which is only correct if k is a constant. So when we were unable to find a constant value of k that meant that our original guess did not work, and we have to try something else.

(d) Now solve this equation correctly using separation of variables.
Solution:

$$\begin{aligned} \frac{dy}{dx} &= y(x + 1) \\ \int \frac{dy}{y} &= \int (x + 1)dx \\ \ln y &= \frac{1}{2}x^2 + x + C \\ y &= Ce^{x^2/2+x} \end{aligned}$$

Final Answer: $y(x) = Ce^{x^2/2+x}$

1.110 $t^2\left(d^2r/dt^2\right) - 9t\,(dr/dt) + 16r = 4$ is an example of a "Cauchy–Euler equation." Such equations appear in a number of physics and engineering applications.

(a) Write the complementary homogeneous equation.
Solution: $t^2(d^2r/dt^2) - 9t(dr/dt) + 16r = 0$

(b) Plug $r = e^{kt}$ into the equation you wrote in Part a. Show that this solution will *not* work for any constant k: this equation has no exponential solution.
Solution: The left side of the complementary equation becomes
$\left(k^2t^2 - 9kt + 16\right)e^{kt}$
We can solve this for k using the quadratic formula, but the resulting k will be a function of t, contradicting our assumption that k is a constant. To put it another way, there is no constant k for which this function is uniformly zero. Therefore $r = e^{kt}$ is not a solution for any constant k.

(c) Plug the guess $r = t^n$ (where n is a constant) into the equation you wrote in Part a. Solve the resulting algebraic equation for n: you should find two solutions.
Solution:

$$\begin{aligned}[n(n-1) - 9n + 16]\,t^n &= 0\\ (n^2 - 10n + 16)t^n &= 0\\ (n-2)(n-8)t^n &= 0\end{aligned}$$

So $n = 2$ or $n = 8$.

(d) Write the general solution to the equation you wrote in Part a.
Solution: $r_C = At^2 + Bt^8$

(e) Find a specific solution to the original (inhomogeneous) equation.
Solution: $r_P = 1/4$

(f) Write the general solution to the inhomogeneous equation.
Final Answer: $r(t) = At^2 + Bt^8 + 1/4$

(g) What is it about this particular equation that made $r = t^n$ work?
Solution: Each time you take the derivative of t^n the power of t goes down by one. But each successive derivative in the original equation was multiplied by one higher power of t. So when we plugged in t^n all the terms ended up at the same power of t.

1.111 $d^2y/dx^2 - 7\,(dy/dx) + 12y = 0$

Solution: Plugging $y = e^{kx}$ into this equation yields:

$$(k^2 - 7k + 12)e^{kx} = 0$$
$$(k - 3)(k - 4)e^{kx} = 0$$

So $k = 3$ or $k = 4$. Because this is a homogeneous equation we can multiply the two resulting solutions by arbitrary constants and combine them.

$$y = Ae^{3x} + Be^{4x}$$

Final Answer: $y = Ae^{3x} + Be^{4x}$

1.112 $d^2s/dt^2 + 3\,(ds/dt) + 7s = 0$

Solution: Plugging $s = e^{kt}$ into this equation yields:

$$(k^2 + 3k + 7)e^{kt} = 0$$

Because we can't factor this, we use the quadratic equation.

$$k = \frac{-3 \pm \sqrt{9 - 28}}{2}$$

The negative discriminant tells us there will be no real exponential solutions.

Final Answer: No real exponential solutions.

1.113 $d^2s/dt^2 + 5\,(ds/dt) - 4s = 0$

Solution: Plugging $s = e^{kt}$ into this equation yields:

$$(k^2 + 5k - 4)e^{kt} = 0$$

Because we can't factor this, we use the quadratic equation.

$$k = \frac{-5 \pm \sqrt{25 + 16}}{2}$$

Because this is a homogeneous equation, we can multiply the resulting solutions by constants and combine them.

$$s = Ae^{-(5+\sqrt{41})t/2} + Bte^{-(5-\sqrt{41})t/2}$$

Final Answer: $s = Ae^{-(5+\sqrt{41})t/2} + Bte^{-(5-\sqrt{41})t/2}$

1.114 $d^2\rho/dz^2 + 10\,(d\rho/dz) + 25\rho = 0$

Solution: Plugging $\rho = e^{kz}$ into this equation yields:

$$(k^2 + 10k + 25)e^{kz} = 0$$
$$(k+5)^2 e^{kz} = 0$$

So $k = -5$ leads to the solution $\rho = e^{-5z}$. We can multiply this solution by one arbitrary constant, but it is still not the general solution because we do not have two arbitrary constants.

Final Answer: $\rho = Ae^{-5z}$ is a solution but not the general solution

1.115 $d^2y/dx^2 + (k^2 + p^2)y = 0$

Solution: This is a simple harmonic oscillator equation.

$$\frac{d^2y}{dx^2} = -\left(k^2 + p^2\right) y$$

The role of ω^2 is played by $k^2 + p^2$, so the solution is:

$$y(x) = A \sin\left(\sqrt{k^2 + p^2}\, x\right) + B \cos\left(\sqrt{k^2 + p^2}\, x\right)$$

To check this take the second derivative.

$$\begin{aligned} y'(x) &= A\sqrt{k^2 + p^2} \cos\left(\sqrt{k^2 + p^2}\, x\right) \\ &\quad - B\sqrt{k^2 + p^2} \sin\left(\sqrt{k^2 + p^2}\, x\right) \\ y''(x) &= -A(k^2 + p^2) \sin\left(\sqrt{k^2 + p^2}\, x\right) \\ &\quad - B(k^2 + p^2) \cos\left(\sqrt{k^2 + p^2}\, x\right) \\ &= -(k^2 + p^2)y(x) \end{aligned}$$

You could also write this solution as $C \cos\left(\sqrt{k^2 + p^2}\, x + \phi\right)$.

Final Answer: $y(x) = A \sin\left(\sqrt{k^2 + p^2}\, x\right) + B \cos\left(\sqrt{k^2 + p^2}\, x\right)$

1.116 $t^2(d^2s/dt^2) + 10t(ds/dt) + 20s = 0$

Solution: An exponential would end up with different kinds of functions in each term (e.g. t^2e^{kt} in the first term and e^{kt} in the third). So would a trig function. Because the powers of t are decreasing with every term, try a guess of the form $s = t^p$.

$p(p-1)t^p + 10pt^p + 20t^p = 0$

$p^2 + 9p + 20 = 0$

The solutions are $p = -4$ and $p = -5$.

$$s(t) = At^{-4} + Bt^{-5}$$

Final Answer: $s(t) = At^{-4} + Bt^{-5}$

1.117 $d^4y/dx^4 - y = 0$

Solution: It may be best to rewrite this as follows.

$$\frac{d^4y}{dx^4} = y$$

What function is its own *fourth* derivative? Both exponential and trig functions work here; after four derivatives they end up where they started, with all negatives eliminated. This gives us the requisite four arbitrary constants.

$$y(x) = A\sin x + B\cos x + Ce^x + De^{-x}$$

Final Answer: $y(x) = A\sin x + B\cos x + Ce^x + De^{-x}$

1.118 $d^2y/dx^2 + \omega^2 y = e^{kx}$ *Hint*: To find a particular solution, guess $y_p = ae^{kx}$ and solve for the constant a.

Solution: Begin with the complementary homogeneous equation:

$$d^2y/dx^2 + \omega^2 y = 0$$

This is the simple harmonic oscillator equation.

$$y_C = A\sin(\omega x) + B\cos(\omega x)$$

Now we need a particular solution to the original equation. The exponential function on the right suggests an exponential guess $y_P = ae^{kx}$.

$$ak^2e^{kx} + a\omega^2e^{kx} = e^{kx}$$

This works if $a = 1/\left(k^2 + \omega^2\right)$.

$$y(x) = A\sin(\omega x) + B\cos(\omega x) + \frac{e^{kx}}{k^2 + \omega^2}$$

Final Answer: $y(x) = A\sin(\omega x) + B\cos(\omega x) + e^{kx}/(k^2 + \omega^2)$

1.119 $x^2(d^2u/dx^2) + 4x(du/dx) - 4u = 4$

Solution: The complementary equation is $x^2(d^2u/dx^2) + 4x(du/dx) - 4u = 0$. Guess $u_C = x^p$.

$$\begin{aligned} p(p-1)x^p + 4px^p - 4x^p &= 0 \\ (p^2 + 3p - 4)x^p &= 0 \\ (p+4)(p-1)x^p &= 0 \end{aligned}$$

So $p = -4$ or $p = 1$ and the complementary solution is $u_C = Ax^{-4} + Bx$. The simplest particular solution is $u = -1$.

$$u(x) = \frac{A}{x^4} + Bx - 1$$

Final Answer: $u(x) = Ax^{-4} + Bx - 1$

1.120 The equation $d^2x/dt^2 + 4(dx/dt) + 3x = \sin t$ can represent a harmonic oscillator that is damped (by friction or air resistance) but also *driven* by an oscillatory force.

(a) Write the complementary homogeneous equation. Find the general solution by assuming an exponential form.
Solution: $d^2x/dt^2 + 4(dx/dt) + 3x = 0$
Plugging $x = e^{kt}$ into the homogeneous differential equation and solving for k yields $k = -1, -3$. Therefore, your two solutions are $x_1 = e^{-t}$ and $x_2 = e^{-3t}$. Multiplying these two solutions by arbitrary constants and adding them together yields the general solution: $x = Ae^{-t} + Be^{-3t}$.

(b) To find a specific solution to the original (inhomogeneous) equation, begin by guessing a solution of the form $x = A\sin t + B\cos t$. The numbers A and B do not represent arbitrary constants here; we are looking for one value that will work. Plug your guess into the differential equation.
Solution: $(-A\sin t - B\cos t) + 4(A\cos t - B\sin t) + 3(A\sin t + B\cos t) = \sin t$

(c) Rearrange the resulting algebraic equation into the form:
(bunch of constants) $\sin t$ + *(different bunch of constants)* $\cos t = \sin t$
Solution: $(-A - 4B + 3A)\sin t + (-B + 4A + 3B)\cos t = \sin t$

(d) To solve this equation, set the first "bunch of constants" to 1 and the second to 0. You can now solve for A and B.
Solution: $A = 1/10$ and $B = -1/5$

(e) Write the general solution to this differential equation.
Final Answer: $x(t) = Ae^{-3t} + Be^{-t} + (1/10)(\sin t - 2\cos t)$

1.121 Consider the differential equation $d^2y/dx^2 - 6\,(dy/dx) + 9y = 3$.

(a) Find all exponential solutions to the complementary homogeneous equation.
Solution: Plugging $y = e^{kx}$ into the complementary equation $y'' - 6y' + 9 = 0$ yields:

$$(k^2 - 6k + 9)e^{kx} = 0$$

Solving, $(k-3)^2 = 0$ so $k = 3$. The solution is therefore e^{3x}. As with any solution to a homogeneous equation we can multiply this by an arbitrary constant.

$$y = Ae^{3x}$$

(b) Find a specific solution to the original equation.
Solution: $y_P = 1/3$

(c) Write a solution to the original differential equation that has an arbitrary constant in it.
Solution: $y = Ae^{3x} + 1/3$

(d) Show that the function $y = xe^{3x}$ is a solution to the complementary homogeneous equation.
Solution: Plugging $y = xe^{3x}$ into the complementary homogeneous equation, you get:
$(6e^{3x} + 9xe^{3x}) - 6(e^{3x} + 3xe^{3x}) + 9(xe^{3x}) = 0$
Combining like terms, you get:
$0e^{3x} + 0xe^{3x} = 0$
So the function satisfies the homogeneous differential equation.

(e) Write the general solution (complete with two independent arbitrary constants) to the original differential equation.
Final Answer: $y = Ae^{3x} + Bxe^{3x} + 1/3$

1.122 The equation $x''(t) - 7x'(t) + 10x = 50 + e^{3t}$ has two inhomogeneous terms. You can solve it by approaching them separately.

(a) Find the general solution $x_c(t)$ to the complementary homogeneous equation.
Solution: The constant coefficients suggest an exponential guess $x = e^{kt}$.

$$(k^2 - 7k + 10)e^{kt} = 0$$
$$(k-2)(k-5)e^{kt} = 0$$

So $k = 2$ or $k = 5$, leading to $x = Ae^{2t} + Be^{5t}$.

(b) Find a particular solution $x_{p1}(t)$ to the equation $x''(t) - 7x'(t) + 10x = 50$. You can do this quickly, by just looking.
Solution: $x_{p1}(t) = 5$

(c) Find a particular solution $x_{p2}(t)$ to the equation $x''(t) - 7x'(t) + 10x = e^{3t}$. To do this, plug in a solution of the form $x = Me^{3t}$ and find the value of M that works.
Solution: $x = Me^{3t} \quad \rightarrow \quad x' = 3Me^{3t} \quad \rightarrow \quad x'' = 9Me^{3t}$
Plugging all that in gives us the equation: $9Me^{3t} - 21Me^{3t} + 10Me^{3t} = e^{3t}$
So $-2M = 1$ or $M = -1/2$. We therefore find $x_{p2}(t) = -(1/2)e^{3t}$.

(d) Write the function $x_c(t) + x_{p1}(t) + x_{p2}(t)$ and show that it solves the original differential equation.
Final Answer: $x(t) = Ae^{2t} + Be^{5t} - (1/2)e^{3t} + 5$

1.123 In the Explanation for slope fields (Section 1.4), we found that one specific solution to the equation $dy/dx = y - 2x$ is $y = 2x + 2$. We discussed the behavior of other solutions, but did not find them analytically.

(a) Begin by rewriting the differential equation in a more standard form, as $a_1(x)(dy/dx) + a_0(x)y = f(x)$.
Solution: $dy/dx - y = -2x$

(b) Write and solve the complementary homogeneous equation.
Solution: The equation is $dy/dx - y = 0$. We can solve this by inspection: $dy/dx = y$ means that y is any function that is its own derivative, so $y_c = Ce^x$.

(c) Find a particular solution to the original equation by guessing a solution of the form $y_p = Ax + B$.
Solution: Plugging this into the original equation yields:

$$A - (Ax + B) = -2x$$
$$-Ax + (A - B) = -2x$$

We rearranged this equation to put both sides in the form of a standard linear equation $y = mx + b$. For both sides to be the *same* line their slopes (i.e., the coefficients of x) must be the same, and their y-intercepts (the constant parts) must also be the same.

$$-A = -2$$
$$A - B = 0$$

Solving, $A = 2$ and $B = 2$, so the solution is $y_p = 2x + 2$. This verifies what we had found earlier.

(d) Find the general solution to the original equation.
Final Answer: $y = Ae^x + 2x + 2$

(e) Test your general solution.
Solution: Plugging the general solution into the original differential equation, you get $(Ae^x + 2) - (Ae^x + 2x + 2) = -2x$. Combining like terms you get $-2x = -2x$, so the general solution satisfies the original differential equation.

(f) In the slope field explanation we concluded that if a function lies perfectly on the line $y = 2x + 2$ then it will stay there forever, but if it does not lie on that line it will move farther and farther away from that line. Explain how your general solution correctly predicts both of these behaviors.
Solution: If a function lies perfectly on the line $y = 2x + 2$ then the constant $A = 0$ in the general solution, so the solution is just $y = 2x + 2$.
If a function does NOT lie perfectly on that line, then the constant A will not be 0, and therefore the exponential term Ae^x will contribute to the solution. Since the exponential term blows up as your x-values increase, then your solutions will get farther and farther away from the line $y = 2x + 2$. If $A > 0$ your function will rise farther and farther above the line; for $A < 0$ your function will fall more and more below the line.

1.124 Consider the equation $y^2 + (dy/dx)^2 = 1$. *Important*: don't confuse $(dy/dx)^2$ with d^2y/dx^2, which is something completely different!

(a) Find the two constant solutions to this equation.
Final Answer: $1, -1$

(b) By experimenting with basic functions, find another solution.
Final Answer: $\sin x$ or $\cos x$

(c) With a little more trial and error, rewrite your answer to Part b with an arbitrary constant in it. Make sure it still solves the differential equation.
Final Answer: $\sin(x + C)$ or $\cos(x + C)$

(d) Is the solution you found in Part c the general solution? Why or why not?
Solution: We normally expect that if we have a solution with one arbitrary constant to a first-order differential equation, we have the general solution. But that is not a hard-and-fast rule for linear differential equations, and non-linear equations (such as this one) are even less reliable.
In this case you know it is not the general solution because there is nothing you can plug in for the constant C that will turn this solution into the constant solutions from the beginning of the problem.
Final Answer: No

1.125 Consider the equation $(dy/dx)^2 = y$. *Important*: don't confuse $(dy/dx)^2$ with d^2y/dx^2, which is something completely different!

(a) Plug in a solution of the form $y = kx^p$ where k and p are constants.

Solution:

$$\begin{aligned} y &= kx^p \\ \frac{dy}{dx} &= kpx^{p-1} \\ \left(\frac{dy}{dx}\right)^2 &= k^2p^2x^{2p-2} \\ k^2p^2x^{2p-2} &= kx^p \end{aligned}$$

(b) You should now have an equation of the form $a_1x^{b_1} = a_2x^{b_2}$. The only way these can be the same function is if $a_1 = a_2$ *and* $b_1 = b_2$. Use these two facts to solve for k and p and write a solution to the differential equation. (You may find the solution $y = 0$; that function does work, but don't use it.)

Solution:

$$\begin{aligned} 2p - 2 &= p \\ k^2p^2 &= k \end{aligned}$$

The first equation solves to $p = 2$ and the second to $k = 1/4$, so the solution is $y = (1/4)x^2$.

Final Answer: $y = (1/4)x^2$

(c) Show that your solution works. Then multiply your solution by an arbitrary constant and show that it *doesn't* work. Explain why this doesn't violate the principle of linear superposition.

Solution:

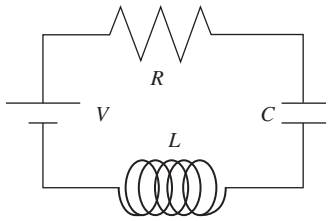
$$y = \frac{1}{4}x^2 \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{2}x \quad \rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^2$$

We end up back at y, so that works.

$$y = \frac{A}{4}x^2 \quad \rightarrow \quad \frac{dy}{dx} = \frac{A}{2}x \quad \rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{A^2}{4}x^2$$

We do *not* end up back at y, so it doesn't work. The differential equation does not follow the principle of linear superposition because it is not linear.

1.126 The picture shows a standard diagram for a circuit with four elements: a battery that maintains a constant voltage V, a resistor with resistance R, a capacitor with capacitance C, and an inductor with inductance L.



A charge Q will build up on the capacitor according to the equation $V = R(dQ/dt) + Q/C + L(d^2Q/dt^2)$.

(a) Find the function $Q(t)$ for a circuit with $V = 9$, $R = 3$, $C = 1/5$, and $L = 1/4$.

Solution: The equation we are solving is:

$$\frac{1}{4}\frac{d^2Q}{dt^2} + 3\frac{dQ}{dt} + 5Q = 9$$
$$\frac{d^2Q}{dt^2} + 12\frac{dQ}{dt} + 20Q = 36$$

For the complementary homogeneous equation, the guess $Q = e^{kt}$ leads to $k^2 + 12k + 20 = 0$ so $k = -2$ or -10. For a particular solution, the constant $Q = 9/5$ suggests itself.

$$Q(t) = Ae^{-2t} + Be^{-10t} + \frac{9}{5}$$

Final Answer: $Q(t) = Ae^{-2t} + Be^{-10t} + 9/5$

(b) Your solution should consist of a "transient" part that dies out quickly and a "steady-state" part that represents the long-term behavior of the circuit. Discuss the steady-state solution: what is the circuit doing? How could you have predicted this behavior—quantitatively, not just qualitatively—without solving the differential equation?

Solution: The steady-state solution is $Q = 9/5$. We could predict this from the differential equation because when Q is a constant, Q' and Q'' are both zero, so the differential equation becomes $9 = (9/5)/(1/5)$. Physically, this solution represents a charge buildup on the capacitor that completely blocks the current. With no current flowing, there is no voltage drop across the resistor or inductor, so the entire voltage drop occurs across the capacitor. That voltage drop is $Q/C = (9/5)/(1/5)$ which is 9, the voltage of the battery.

1.127 Newton's law of gravity says that an object in space falling directly towards Earth will experience an acceleration $r''(t) = -GM_E/r^2$, where G and M_E are constants representing the strength of gravity and the mass of the Earth, and r is the object's distance from the center of the Earth. (We assume here that the object's mass is much smaller than the Earth's.)

(a) Try a power law solution of the form $r(t) = a(t-b)^p$ and plug it in. As always, the result should be an algebraic equation.
Final Answer: $ap(p-1)(t-b)^{p-2} = -GM_E(t-b)^{-2p}/a^2$

(b) The equation you found in Part a should have been in the form $something(t-b)^{something} = something(t-b)^{something}$. The only way the two sides of that equation can represent the same function is if they both have the same exponent *and* the same coefficient in front, so you can rewrite this as two algebraic equations and find the values of two of the constants in your original guess. Write down the solution $r(t)$ that you find and verify that it does solve the original differential equation.
Solution:

$$p - 2 = -2p$$
$$ap(p-1) = -GM_E/a^2$$

The first equation becomes $p = 2/3$ and the second then becomes $a = \sqrt[3]{9GM_E/2}$.

$$r(t) = \sqrt[3]{\frac{9GM_E}{2}}(t-b)^{2/3}$$
$$r'(t) = \frac{2}{3}\sqrt[3]{\frac{9GM_E}{2}}(t-b)^{-1/3}$$
$$r''(t) = -\frac{2}{9}\sqrt[3]{\frac{9GM_E}{2}}(t-b)^{-4/3}$$
$$-\frac{GM_E}{r^2} = -GM_E\left(\frac{2}{9GM_E}\right)^{2/3}(t-b)^{-4/3}$$

The two are the same, so the solution works.
Final Answer: $r(t) = (9GM_E/2)^{1/3}(t-b)^{2/3}$

(c) Does your solution have any arbitrary constants in it? Is it the general solution?
Final Answer: Yes, and no

1.128 In the Explanation (Section 1.6.2) we showed that a damped harmonic oscillator is modeled by the equation $ma = -cv - kx$. We then solved this equation for particular values of the constants—values that were chosen to work out nicely. In this problem you will take up the challenge more generally.

(a) Explain why the nature of a spring dictates that k must be positive, and the nature of a damping force dictates that c must be positive.

Solution: The force of the spring acts opposite the direction of displacement from the equilibrium. So the spring force, $-kx$, and the displacement, x, must be opposite in sign. A similar argument holds true for the damping force, which acts opposite the direction of velocity v.

(b) Plug the guess $x = e^{pt}$ into the differential equation $m(d^2x/dt^2) + c(dx/dt) + kx = 0$.

Final Answer: $(mp^2 + cp + k)e^{pt} = 0$

(c) Solve the resulting equation for p.

Final Answer: $p = -c/(2m) \pm \sqrt{c^2 - 4mk}/(2m)$

(d) If the resulting equation has only one solution for p, the system is said to be "critically damped." What relationship between the constants c, k, and m leads to a critically damped oscillator? Express this relationship by writing a formula for c as a function of k and m.

Solution: For there to be only one solution, the square root term must be 0. This implies that $c^2 - 4mk = 0$. Solving for c, you get $c = \sqrt{4mk}$.

Final Answer: $c = \sqrt{4mk}$

(e) If c is greater than the formula you found in Part d, the system is said to be "overdamped." How many real p-values do you find in this case, and are they positive or negative? What sort of behavior results from an overdamped system?

Solution: If $c > \sqrt{4mk}$, then $c^2 - 4mk > 0$. Since the expression under the square root is positive and greater than 0, you will have two real solutions. Subtracting the square root term gives you a negative p value. Adding the square root term also gives you a negative p value, because you know that $c^2 - 4mk < c^2$, so $\sqrt{c^2 - 4mk} < \sqrt{c^2} = c$. Thus, $-c + \sqrt{c^2 - 4mk} < 0$. So both p-values will be real and negative, and your general solution will be the sum of two decaying exponentials. This means that your spring will asymptotically approach equilibrium without oscillating.

(f) If c is less than the formula you found in Part d, how many real p-values do you find? What does that suggest about our guess? About the behavior of the system?

Solution: If $c < \sqrt{4mk}$, then $c^2 - 4mk < 0$. The expression under the square root sign is negative, which doesn't give you any real values for p. This suggests that solutions of the form e^{pt} do not work for any real values of c. The spring will not just decay down to zero; it will oscillate.

You can solve in such cases by assuming a solution that oscillates with a decaying amplitude (as you would expect from the physics of the situation). If you are comfortable with complex numbers you can solve more easily with complex exponential solutions.

1.129 Use Equation 1.6.4 to determine if each of the differential equations below is linear, and if so whether it is homogeneous. Then test the proposed solutions y_1 and y_2, and also their sum $y_1 + y_2$.

(a) $y'' - y = 0, y_1 = e^x, y_2 = e^{-x}$.

Solution: Each term is just y or one of its derivatives times a constant (1 or -1), so this is a linear homogeneous equation.

$$y_1'' - y_1 = e^x - e^x = 0$$
$$y_2'' - y_2 = e^{-x} - e^{-x} = 0$$
$$(y_1 + y_2)'' - (y_1 + y_2) = (e^x + e^{-x}) - (e^x + e^{-x}) = 0$$

The two proposed solutions work, and so does their sum.

Final Answer: The equation is linear and homogeneous and all three solutions work.

(b) $y'' - y = 2\sin x, y_1 = e^x, y_2 = -\sin x$.

Solution: Each term that has y in it is just y or one of its derivatives times a constant (1 or -1), so this is a linear equation. Since it has a term without y it is inhomogeneous.

$$y_1'' - y_1 = e^x - e^x = 0$$
$$y_2'' - y_2 = \sin x + \sin x = 2\sin x$$
$$(y_1 + y_2)'' - (y_1 + y_2) = (e^x + \sin x) - (e^x - \sin x) = 2\sin x$$

The first solution doesn't work because it gives zero, but y_2 and the sum $y_1 + y_2$ both work.

Final Answer: The equation is linear and inhomogeneous; y_2 and $y_1 + y_2$ work, but y_1 does not.

(c) $y'' - yy' = 0, y_1 = -2/x, y_2 = 2.$

Solution: This is not a linear homogeneous equation because of the term yy'.

$$y_1'' - y_1 y_1' = -\frac{4}{x^3} - \left(-\frac{2}{x}\right)\left(\frac{2}{x^2}\right) = 0$$
$$y_2'' - y_2 y_2' = 0 - (2)(0) = 0$$
$$(y_1 + y_2)'' - (y_1 + y_2)(y_1 + y_2)' = -\frac{4}{x^3} - \left(-\frac{2}{x} + 2\right)\left(\frac{2}{x^2}\right) = -\frac{4}{x^2}$$

The proposed solutions y_1 and y_2 work, but this time their sum does not because of the cross-term that comes from multiplying y by y'.

These examples confirmed the rule that linear homogeneous differential equations obey the principle of linear superposition, while other equations do not.

Final Answer: The equation is non-linear. The solutions y_1 and y_2 work but their sum does not.

1.130 In this problem you will prove the principle of linear superposition. Because we want a general proof, we begin with the general form for every linear homogeneous differential equation:

$$a_n(x)\frac{d^n y}{dx^n} + \ldots + a_3(x)\frac{d^3 y}{dx^3} + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (1.6.11)$$

(a) Suppose $y = u_1(x)$ is a solution to Equation 1.6.11. Show that the function $y = Au_1(x)$ also works as a solution. (Remember that if A is a constant, $(d/dx)Au_1(x) = A(du_1/dx)$.)
Solution: Taking all the derivatives of $y = Au_1(x)$, you see that the constant A passes through the derivatives to the outside:
$(d/dx)Au_1(x) = A(du_1/dx)$
$(d^2/dx^2)Au_1(x) = A(d^2u_1/dx^2)$
$(d^n/dx^n)Au_1(x) = A(d^nu_1/dx^n)$
Therefore, when you plug in $y = Au_1(x)$ into the differential equation, A will appear as a constant in front of every term and you can factor it out to get:

$$A\left(a_n(x)\frac{d^n u_1}{dx^n} + \ldots + a_3(x)\frac{d^3 u_1}{dx^3} + a_2(x)\frac{d^2 u_1}{dx^2} + a_1(x)\frac{du_1}{dx} + a_0(x)u_1\right) = 0$$

Since you already know that $y = u_1(x)$ satisfies the homogeneous differential equation, everything inside the parenthesis is 0. Therefore, you get $A(0) = 0$. So $y = Au_1(x)$ satisfies the homogeneous equation and is therefore a solution.

(b) Suppose $u_1(x)$ and $u_2(x)$ are both solutions to Equation 1.6.11. Show that the function $u_1(x) + u_2(x)$ also works as a solution.
Solution: When you apply any derivative to the sum $u_1(x) + u_2(x)$, you can apply that derivative to each term individually and then sum up the resulting derivatives. (Additive property of derivatives)
$(d/dx)(u_1(x) + u_2(x)) = (du_1/dx) + (du_2/dx)$
$(d^2/dx^2)(u_1(x) + u_2(x)) = (d^2u_1/dx^2) + (d^2u_2/dx^2)$
$(d^n/dx^n)(u_1(x) + u_2(x)) = (d^nu_1/dx^n) + (d^nu_2/dx^n)$
Plugging in $y = u_1(x) + u_2(x)$ into the homogeneous differential equation and then grouping all the terms with u_1 together and all the terms with u_2 together, you get:

$$\left(a_n(x)\frac{d^n u_1}{dx^n} + \ldots + a_1(x)\frac{du_1}{dx} + a_0(x)u_1\right) + \left(a_n(x)\frac{d^n u_2}{dx^n} + \ldots + a_1(x)\frac{du_2}{dx} + a_0(x)u_2\right) = 0$$

Since $y = u_1$ and $y = u_2$ are solutions by themselves, they both satisfy the homogeneous differential equation individually. Therefore, the quantities in both parentheses are 0, and you get 0+0=0. So $y = u_1(x) + u_2(x)$ satisfies the homogeneous differential equation and is therefore a solution.

(c) Now consider a linear *inhomogeneous* differential equation:

$$a_n(x)\frac{d^n y}{dx^n} + \ldots + a_3(x)\frac{d^3 y}{dx^3} + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \quad (f(x) \neq 0) \tag{1.6.12}$$

Suppose $u_h(x)$ is a solution to Equation 1.6.11 and $u_i(x)$ is a solution to Equation 1.6.12. Show that the function $u_h(x) + u_i(x)$ solves Equation 1.6.12 but not 1.6.11.

Solution: By the same principle as in part (b), $(d^n/dx^n)(u_h(x) + u_i(x)) = (d^n u_h/dx^n) + (d^n u_i/dx^n)$ for any n-value of the derivative. Plugging $y = u_h(x) + u_i(x)$ into the inhomogeneous differential equation and grouping all the terms with u_h together and all the terms with u_i together, you get:

$$\left(a_n(x)\frac{d^n u_h}{dx^n} + \ldots + a_1(x)\frac{du_h}{dx} + a_0(x)u_h\right) + \left(a_n(x)\frac{d^n u_i}{dx^n} + \ldots + a_1(x)\frac{du_i}{dx} + a_0(x)u_i\right) = f(x)$$

Since $y = u_h(x)$ satisfies the homogeneous equation, the quantity in the left parenthesis is 0. Since $y = u_i(x)$ satisfies the inhomogeneous equation, the quantity in the right parenthesis is $f(x)$. Therefore, your equation reduces to $0 + f(x)$ on the left side. This proves that plugging in the function $u_h(x) + u_i(x)$ gives the result $f(x)$, not zero, so it satisfies the inhomogeneous equation but not the homogeneous.

1.131 To see why the principle of linear superposition only applies to linear equations, consider a non-linear equation of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y^2 = 0 \tag{1.6.13}$$

(a) Suppose $u_1(x)$ and $u_2(x)$ are both solutions to Equation 1.6.13. Show that the function $u_1(x) + u_2(x)$ *does not work* as a solution.
Solution: Plugging in $u_1(x) + u_2(x)$ into this non-linear differential equation and using the additive property of derivatives to write each derivative of the sum as the sum of the derivatives on each function, you will end up with the following equation:

$$a_2(x)\left(\frac{d^2u_1}{dx^2} + \frac{d^2u_2}{dx^2}\right) + a_1(x)\left(\frac{du_1}{dx} + \frac{du_2}{dx}\right) + a_0(x)\left(u_1^2 + 2u_1u_2 + u_2^2\right) = 0$$

Following the same method as you did in Problem 21(b), you group all the terms with u_1 together and all the terms with u_2 together:

$$\left(a_2(x)\frac{d^2u_1}{dx^2} + a_1(x)\frac{du_1}{dx} + a_0(x)u_1\right) + \left(a_2(x)\frac{d^2u_2}{dx^2} + a_1(x)\frac{du_2}{dx} + a_0(x)u_2\right) + 2u_1u_2 = 0$$

You notice that there is one term, $2u_1u_2$, which cannot be grouped because it contains both variables. However, the quantities in the left and the right hand parenthesis can be reduced to 0, since both u_1 and u_2 are solutions and satisfy the differential equation individually. Therefore this solution does not work unless u_1 or u_2 happens to be zero.

(b) Repeat Part (a) assuming that dy/dx is squared but y is not. Once again assume that $u_1(x)$ and $u_2(x)$ are solutions to this new equation and show that $u_1(x) + u_2(x)$ is not.
Solution: Plugging in $u_1(x) + u_2(x)$ into the differential equation and following the same process as in part (a), you will end up with:

$$a_2(x)\left(\frac{d^2u_1}{dx^2} + \frac{d^2u_2}{dx^2}\right) + a_1(x)\left(\left(\frac{du_1}{dx}\right)^2 + 2\left(\frac{du_1}{dx}\right)\left(\frac{du_2}{dx}\right) + \left(\frac{du_2}{dx}\right)^2\right) + a_0(x)\left(u_1 + u_2\right) = 0$$

Grouping all the terms with u_1 together and all the terms with u_2 together, you will have:

$$\left(a_2(x)\frac{d^2u_1}{dx^2} + a_1(x)\left(\frac{du_1}{dx}\right)^2 + a_0(x)u_1\right) + \left(a_2(x)\frac{d^2u_2}{dx^2} + a_1(x)\left(\frac{du_2}{dx}\right)^2 + a_0(x)u_2\right) + 2a_1\left(\frac{du_1}{dx}\right)\left(\frac{du_2}{dx}\right) = 0$$

The quantities in the first and second parentheses can be reduced to 0, since both u_1 and u_2 are solutions and satisfy the differential equation individually. Therefore, on the left hand side you will have the mixed derivative term remaining and on the right hand side you will have 0. If u_1 and u_2 are both independent solutions to the differential equation, then this mixed term will, in general, not be equal to 0. Therefore, the sum $u_1(x) + u_2(x)$ does not satisfy the differential equation.

1.132 Consider the linear differential equation $x'(t) = x(t)$.

(a) Show that $x(t) = |C|e^t$ is a solution for any value of the constant C.

Solution: The derivative of $|C|e^t$ is $|C|e^t$ so there you are.

(b) Since this is a first-order linear differential equation and we have a solution with an arbitrary constant you would normally expect it to be the general solution. Prove that it isn't by finding a solution that doesn't match this for any value of C.

Solution: $x(t) = -e^t$

The moral of the story is that it isn't always easy to tell if you have the "general solution" or not. But the truth is, we really had to contrive that example. In practice if you have n arbitrary constants for an nth order linear differential equation, you almost certainly have the general solution. (Be much more careful making generalizations about non-linear equations.)

1.133 In this problem you will show that the principle of linear superposition does not apply to the non-linear differential equation $y'' - y'y = 0$.

(a) What makes this equation non-linear?
Final Answer: $y'y$

(b) Assume you have found two solutions y_1 and y_2. Plug the guess $y = y_1 + y_2$ into the left side of this equation and simplify as much as possible using the fact that y_1 and y_2 are both solutions. What term are you left with that doesn't go away?
Solution:

$$\begin{aligned}(y_1 + y_2)'' - (y_1 + y_2)'(y_1 + y_2) &= 0\\ y_1'' + y_2'' - y_1'y_1 - y_1'y_2 - y_2'y_1 - y_2'y_2 &= 0\\ (y_1'' - y_1'y_1) + (y_2'' - y_2'y_2) - y_1'y_2 - y_2'y_1 &= 0\end{aligned}$$

Because y_1 is a solution to the differential equation, the first parenthetical expression $y_1'' - y_1'y_1$ must equal zero. The second parenthetical expression similarly vanishes. But that leaves the terms $y_1'y_2$ and $y_2'y_1$ which are *not* guaranteed to be zero, so we do not necessarily have a solution.

(c) Show that $y_1 = 2\tan x$ is a solution.
Solution:

$$y_1 = \frac{2\sin x}{\cos x} \quad \rightarrow \quad y_1' = \frac{2}{\cos^2 x} \quad \rightarrow \quad y_1'' = \frac{4\sin x}{\cos^3 x}$$

Let's plug all that into the left side of the differential equation and see what we get.

$$\frac{4\sin x}{\cos^3 x} - \left(\frac{2}{\cos^2 x}\right)\left(\frac{2\sin x}{\cos x}\right) = 0$$

It works!

(d) Show that $y_2 = 2$ is a solution.
Solution:

$$y_2 = 2 \quad \rightarrow \quad y_2' = y_2'' = 0$$

Let's plug all that into the left side of the differential equation and see what we get.

$$0 - (0)(2) = 0$$

It works!

(e) Plug in $y = 2\tan x + 2$. Show that it doesn't work, and show that the term you are left with is the same term you found in Part b.
Solution:

$$y_1 = \frac{2\sin x}{\cos x} + 2 \quad \rightarrow \quad y_1' = \frac{2}{\cos^2 x} \quad \rightarrow \quad y_1'' = \frac{4\sin x}{\cos^3 x}$$

Let's plug all that into the left side of the differential equation and see what we get.

$$\frac{4\sin x}{\cos^3 x} - \left(\frac{2}{\cos^2 x}\right)\left(\frac{2\sin x}{\cos x} + 2\right) = -\frac{4}{\cos^2 x}$$

We don't get zero this time, so it is not a solution. Our previous solution suggested that we would be left with this.

$$-y_1'y_2 - y_2'y_1 = -\left(\frac{2}{\cos^2 x}\right)(2) - (0)\left(\frac{2\sin x}{\cos x}\right) = -\frac{4}{\cos^2 x}$$

Sure enough.

1.134 Consider the equation $dy/dx = 1/y$.

(a) Explain how you can tell this equation is non-linear.

Solution: A linear equation cannot have $1/y$.

(b) Show that $y = \sqrt{2x + C}$ is a solution.

Solution:

$$y(x) = (2x + C)^{1/2}$$
$$y'(x) = \frac{1}{2}(2x + C)^{-1/2} \times 2 = \frac{1}{\sqrt{2x + C}} = \frac{1}{y}$$

We have a solution with an arbitrary constant for a first-order equation, so it must be the general solution…right? That's a good rule of thumb but it does not always work, especially for non-linear equations.

(c) Show that $y = -\sqrt{2x + 1}$ is a solution, and that it cannot be written in the form given by Part b. (This shows that Part b was not in fact the general solution.)

Solution:

$$y(x) = -(2x + 1)^{1/2}$$
$$y'(x) = -\frac{1}{2}(2x + 1)^{-1/2} \times 2 = -\frac{1}{\sqrt{2x + 1}} = \frac{1}{y}$$

But the solution given in the first part was positive for any value of C, while this function is negative.

1.135 $x'(t) = y$, $y'(t) = x$. $x(t) = Ae^t$, $y(t) = Ae^t$

Solution:

$$\begin{aligned} x = Ae^t \quad &\rightarrow \quad x'(t) = Ae^t = y(t) \\ y = Ae^t \quad &\rightarrow \quad y'(t) = Ae^t = x(t) \end{aligned}$$

So it is a solution. Note that it is not the general solution, which would require two arbitrary constants.

Final Answer: solution

1.136 $x'(t) = -x + y$, $y'(t) = x - y$. $x(t) = A(1 + e^{-2t}) + B(1 - e^{-2t})$, $y(t) = A(1 - e^{-2t}) + B(1 + e^{-2t})$

Solution:

$$\begin{aligned} x(t) &= A(1 + e^{-2t}) + B(1 - e^{-2t}) \\ x'(t) &= -2Ae^{-2t} + 2Be^{-2t} \\ -x + y &= -2Ae^{-2t} + 2Be^{-2t} \end{aligned}$$

Looks good so far!

$$\begin{aligned} y(t) &= A(1 - e^{-2t}) + B(1 + e^{-2t}) \\ y'(t) &= 2Ae^{-2t} - 2Be^{-2t} \\ x - y &= 2Ae^{-2t} - 2Be^{-2t} \end{aligned}$$

So it works. Because both equations are first- order linear we should have two arbitrary constants, and we do, so it is the general solution.

Final Answer: General solution

1.137 $x'(t) = 2y, y'(t) = x + y.\ x(t) = Ae^{2t} + 2Be^{-t}, y(t) = Ae^{2t} - Be^{-t}$

Solution:

$$x(t) = Ae^{2t} + 2Be^{-t}$$
$$x'(t) = 2Ae^{2t} - 2Be^{-t}$$
$$2y = 2Ae^{2t} - 2Be^{-t}$$

Looks good so far!

$$y(t) = Ae^{2t} - Be^{-t}$$
$$y'(t) = 2Ae^{2t} + Be^{-t}$$
$$x + y = 2Ae^{2t} + Be^{-t}$$

So it works. Because both equations are first- order linear we should have two arbitrary constants, and we do, so it is the general solution.

Final Answer: solution

1.138 $x'(t) = 2xy$, $y'(t) = x^2 + y^2$. $x(t) = 1/(t^2 - 1)$, $y(t) = -t/(t^2 - 1)$ *Hint*: The equations are non-linear, but in this case you should still be able to figure out if this solution is general.

Solution:

$$x(t) = (t^2 - 1)^{-1}$$
$$x'(t) = -(t^2 - 1)^{-2}(2t) = -\frac{2t}{(t^2 - 1)^2}$$
$$2xy = -\frac{2t}{(t^2 - 1)^2}$$

So far so good!

$$y(t) = -\frac{t}{t^2 - 1}$$
$$y'(t) = \frac{t^2 + 1}{(t^2 - 1)^2}$$
$$x^2 + y^2 = \frac{1 + t^2}{(t^2 - 1)^2}$$

So we have a solution. Note, however, that this solution has no arbitrary constants at all, so it has no flexibility for meeting initial conditions. For instance, at $t = 0$ this solution must be at the point $(x, y) = (-1, 0)$.

Final Answer: solution

1.139 $x''(t) = y, y''(t) = -x.\ x(t) = Ae^t, y(t) = Ae^t$

Solution:

$$x = Ae^t \quad \rightarrow \quad x' = Ae^t \quad \rightarrow \quad x'' = Ae^t = y$$

So that looks good! But…

$$y = Ae^t \quad \rightarrow \quad y' = Ae^t \quad \rightarrow \quad y'' = Ae^t \neq -x$$

So this is not a solution.

Final Answer: Not a solution

1.140 Find the general solutions (two independent arbitrary constants) to the following sets of coupled equations. You should be able to do these mostly by inspection, but you can decouple them if you get stuck.

(a) $f'(x) = g(x), g'(x) = f(x)$

Solution: As the problem said, you can approach these by decoupling. But we just thought about them for a while.

The most obvious solution to this first part is $f(x) = g(x) = e^x$. A bit less obvious is $f(x) = e^{-x}$ and $g(x) = -e^{-x}$. So we can write the general solution like this.

$$f(x) = Ae^x + Be^{-x}$$
$$g(x) = Ae^x - Be^{-x}$$

If you are familiar with the hyperbolic sine and cosine you can solve this even more simply with them. The hyperbolic sine and cosine are each other's derivatives, with no negative sign; that is part of what makes them useful.

$$f(x) = A\sinh x + B\cosh x$$
$$g(x) = A\cosh x + B\sinh x$$

Final Answer: $f(x) = Ae^x + Be^{-x}$ and $g(x) = Ae^x - Be^{-x}$, or $f(x) = A\sinh x + B\cosh x, g(x) = A\cosh x + B\sinh x$

(b) $f'(x) = -g(x), g'(x) = -f(x)$

Solution: Once you've seen the solution to the previous problem, it helps a bit in thinking through this one. The easier solution to find is $f(x) = g(x) = e^{-x}$. The less obvious one is $f(x) = e^x$ and $g(x) = -e^x$.

$$f(x) = Ae^{-x} + Be^x$$
$$g(x) = Ae^{-x} - Be^x$$

The hyperbolic sine and cosine version of this solution looks like this.

$$f(x) = A\sinh(x) + B\cosh(x)$$
$$g(x) = -A\cosh(x) - B\sinh(x)$$

Final Answer: $f(x) = Ae^{-x} + Be^x$ and $g(x) = Ae^{-x} - Be^x$, or $f(x) = A\sinh(x) + B\cosh(x), g(x) = -A\cosh(x) - B\sinh(x)$

(c) $f'(x) = g(x), g'(x) = 4f(x)$

Solution: A little informal decoupling can be helpful here. When you take the derivative of f you get g; when you take the derivative again you get $4f$; so the second derivative of f is $4f$ and we know the solution to that is $f = e^{2x}$. That means $g = 2e^{2x}$ and we can readily see that it works. A bit more thought leads to the other solution, which looks the same but with -2 instead of 2.

$$f(x) = Ae^{2x} + Be^{-2x}$$
$$g(x) = 2Ae^{2x} - 2Be^{-2x}$$

Final Answer: $f(x) = Ae^{2x} + Be^{-2x}, g(x) = 2Ae^{2x} - 2Be^{-2x}$

1.141 The superstates of Oceania and Eastasia have a complicated ever-changing relationship. For each scenario below write a set of coupled differential equations that might represent their populations $O(t)$ and $E(t)$. Then describe—using equations or words—how you would expect their populations to evolve over time.

(a) Oceania and Eastasia are mutually supportive allies. The greater their combined (total) population, the more their individual populations grow.
Final Answer: $dO/dt = k_1(O + E)$, $dE/dt = k_2(O + E)$

(b) Oceania and Eastasia are at war. The greater the population of Oceania, the more Eastasians are killed each year, and vice versa.
Final Answer: $dO/dt = -k_1E$, $dE/dt = -k_2O$

(c) Oceania and Eastasia are at war, but a different kind of war. If Oceania has more people than Eastasia, then the population of Oceania will grow while the population of Eastasia shrinks. The reverse if Eastasia outnumbers Oceania. The greater the difference in population, the greater the changes.
Final Answer: $dO/dt = k_1(O - E)$, $dE/dt = k_2(E - O)$

(d) Oceania and Eastasia have a treaty of peace and mutual support, but Oceania cheats. The greater the population of Eastasia, the more Oceania thrives; the greater the population of Oceania, the more Eastasia suffers.
Final Answer: $dO/dt = k_1E$, $dE/dt = -k_2O$

1.142 A vat contains a molecules of substance A and b molecules of substance B. Each second, kab reactions occur, each of which turns one molecule of A and two molecules of B into a molecule of C.

(a) Write differential equations for the number of molecules of a, b, and c.
Final Answer: $a' = -kab$, $b' = -2kab$, $c' = kab$

(b) Describe in words how you would expect these numbers to evolve over time.
Solution: a will decrease quickly at first, then more slowly, approaching zero—like an exponential decay. b will do the same, but always decreasing twice as fast as a. c will increase quickly at first and then more slowly, asymptotically approaching the value where all the a and b have been converted.

(c) Now assume that in addition to the chemical reaction described above, 10^{23} molecules of A are being added to the vat each second. Write the new differential equations and discuss how this will change the results over time.
Solution: $a' = -kab + 10^{23}$, $b' = -2kab$, $c' = kab$. As the reactions slow down and the amount of b approaches zero, the amount of c will still approach the same value it did before, leaving a climbing linearly.
Final Answer: $a' = -kab + 10^{23}$, $b' = -2kab$, $c' = kab$

1.143 Tanks A and B with volumes V_A and V_B are both filled with a mixture of water and brine. Pure brine is being poured into tank A at a rate of r gallons per minute. Meanwhile the water/brine mix from tank A is being poured into tank B at the same rate, and the mix from tank B is being poured into the nearby river at the same rate. Since the rates are all the same the volume of liquid in each tank stays the same, and you can assume that the tanks are well mixed, so the fraction of brine leaving each tank equals the total fraction of brine in the tank at that moment.

(a) Write a pair of coupled differential equations for the number of gallons of brine in each tank, G_A and G_B.
Final Answer: $dG_A/dt = r(1 - G_A/V_A)$, $dG_B/dt = r(G_A/V_A - G_B/V_B)$

(b) Physically, what would you expect to happen to G_A and G_B in the long run, and why?
Solution: Both will asymptotically approach being filled with pure brine. $\lim_{t\to\infty} G_A = V_A$ and $\lim_{t\to\infty} G_B = V_B$.

(c) Mathematically, how can you look at the differential equations you wrote and confirm that the physical behavior you described is what G_A and G_B will do at late times? *Hint*: Consider what must be true of G_A and G_B in order for $G_A' = G_B' = 0$, and what will happen to G_A and G_B when that condition isn't met.
Solution: As long as $G_A < V_A$ the formula for G_A' will be positive, so G_A' will keep increasing. When $G_A = V_A$ then $G_A' = 0$; that's the equilibrium value.
The situation is subtler with the G_B equation, but we have just concluded that the ratio G_A/V_A will approach 1 in the long term. When it is roughly 1, then the G_B equation looks exactly the same.

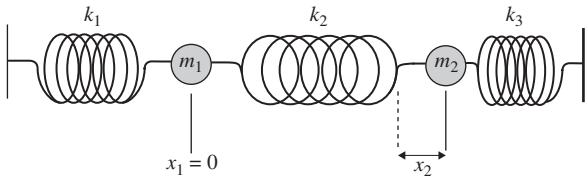
1.144 Tanks A and B, with volumes V_A and V_B, are both filled with a mixture of water and brine. Each minute r gallons of liquid flow from A to B and an equal amount flows from B to A. Since the rates are the same the volume of liquid in each tank stays the same, and you can assume that the tanks are well mixed, so the fraction of brine leaving each tank equals the total fraction of brine in the tank at that moment.

(a) Write a pair of coupled differential equations for the number of gallons of brine in each tank, G_A and G_B.
Final Answer: $dG_A/dt = r(G_B/V_B - G_A/V_A)$, $dG_B/dt = r(G_A/V_A - G_B/V_B)$

(b) Physically, what would you expect to happen to G_A and G_B in the long run, and why?
Solution: Let's suppose, arbitrarily, that A has a higher concentration of brine than B. Then A will be dumping out more brine than it is gaining, and B of course will be gaining more than it dumps. So the concentration of A will go down and the concentration of B will go up.
By this kind of argument you can see that they will approach having exactly the same concentration of brine. When they reach that point they will be exchanging identical streams, so nothing will change.

(c) Mathematically, how can you look at the differential equations you wrote and confirm that the physical behavior you described is what G_A and G_B will do at late times? *Hint*: Consider what must be true of G_A and G_B in order for $G_A' = G_B' = 0$, and what will happen to G_A and G_B when that condition isn't met.
Solution: If A has a higher concentration of brine than B, then $G_A/V_A > G_B/V_B$. This makes the formula for G_A' come out negative and G_B' come out positive. The equilibrium state, when both derivatives are zero, comes when $G_A/V_A = G_B/V_B$.

1.145 The figure shows two balls connected to each other and to the walls by three springs. Ball 1 has mass m_1 and is displaced by an amount x_1 from its equilibrium position, and similarly for Ball 2.



(a) The position $x_1 = x_2 = 0$ represents the equilibrium position. Now imagine that Ball 1 is at this position precisely, but Ball 2 is slightly to the right of this position, as shown above. Which springs now exert force on Ball 1, and in which directions? Which springs now exert force on Ball 2, and in which directions?

Final Answer: Spring 2 pulls m_1 to the right. Springs 2 and 3 pull m_2 to the left.

(b) Now imagine that Ball 1 is displaced to the right by a distance x_1, and Ball 2 is displaced to the right by x_2.

i. The force of Spring 1 on Ball 1 depends only on the position x_1 (the other position is irrelevant). Write a formula for this force.

Solution: $F = -k_1 x_1$

ii. How much is Spring 2 stretched? Your answer should be a function of x_1 and x_2. For example, if both balls are displaced the same amount to the right then Spring 2 isn't stretched at all. As a check on your answer, make sure it gives a positive answer when Spring 2 is longer than its equilibrium length, a negative answer when it is shorter, and 0 when Spring 2 is at its equilibrium length (neither stretched nor compressed.)

Solution: $x_2 - x_1$

iii. Using your answer to Part b, write a formula for the force of Spring 2 on Ball 1.

Solution: $F = k_2(x_2 - x_1)$

iv. Putting the two forces together and using $F = ma$, write a differential equation for $x_1(t)$. Be sure you have the sign of each force correct.

Final Answer: $x_1''(t) = -(k_1/m_1)x_1 + (k_2/m_1)(x_2 - x_1)$

(c) Repeat Part b for x_2.

Final Answer: $x_2''(t) = (k_2/m_2)(x_1 - x_2) - (k_3/m_2)x_2$

You will solve this particular set of coupled differential equations twice: using matrices in Chapter 6, and using Laplace transforms in Chapter 10.

1.146 *[This problem depends on Problem 145.]*

(a) Write the coupled differential equations for Problem 145 for the case where $k_1 = 1$, $k_2 = 4$, $k_3 = 28$, $m_1 = 1$, and $m_2 = 4$.

Solution:

$$\frac{d^2x_1}{dt^2} = -x_1 + 4(x_2 - x_1)$$
$$\rightarrow \frac{d^2x_1}{dt^2} = -5x_1 + 4x_2$$
$$4\frac{d^2x_2}{dt^2} = 4(x_1 - x_2) - 28x_2$$
$$\rightarrow \frac{d^2x_2}{dt^2} = x_1 - 8x_2$$

Final Answer: $d^2x_1/dt^2 = -5x_1 + 4x_2, d^2x_2/dt^2 = x_1 - 8x_2$

(b) The general solution to this problem would have four arbitrary constants. What are the four initial conditions you would need to find them?

Solution: The initial positions and initial velocities of each ball.

Final Answer: $x_1(0), x_2(0), x_1'(0), x_2'(0)$

(c) Verify that $x_1 = 4\cos(2t), x_2 = \cos(2t)$ is one valid solution.

Solution:

$$x_1 = 4\cos(2t)$$
$$x_1'' = -16\cos(2t)$$
$$-5x_1 + 4x_2 = -16\cos(2t)$$

So far so good!

$$x_2 = \cos(2t)$$
$$x_2'' = -4\cos(2t)$$
$$x_1 - 8x_2 = -4\cos(2t)$$

(d) Verify that $x_1 = \sin(3t), x_2 = -\sin(3t)$ is one valid solution.

Solution:

$$x_1 = \sin(3t)$$
$$x_1'' = -9\sin(3t)$$
$$-5x_1 + 4x_2 = -9\sin(3t)$$

So far so good!

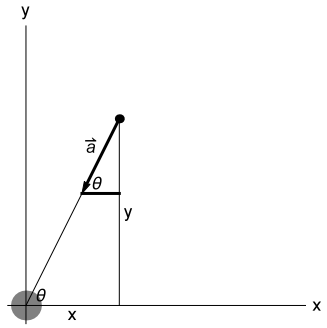
$$x_2 = -\sin(3t)$$
$$x_2'' = 9\sin(3t)$$
$$x_1 - 8x_2 = 9\sin(3t)$$

So it works.

1.147 Define a coordinate system with the sun at the origin and the Earth's position given by (x, y). (The Earth orbits in a plane, so we can ignore the third direction.) We will for simplicity assume the sun doesn't move.

(a) The gravitational acceleration of a planet being pulled on by the sun has magnitude GM/r^2, where G is a constant, M is the sun's mass, and r is the distance between the two objects. Write down the magnitude of the gravitational acceleration of the Earth in terms of x and y.
Final Answer: $GM/(x^2 + y^2)$

(b) Using the fact that the gravitational acceleration points towards the origin (the sun), find the x and y-components of the Earth's gravitational acceleration. *Hint*: You may find it helpful to draw a picture and label an angle θ, but your final answer should be in terms of x and y, not θ.
Solution:



From the drawing we can see that $|a_x| = |\vec{a}| \cos\theta$. But we can also see that $\cos\theta = x/\sqrt{x^2 + y^2}$. Similar logic applies to a_y. Finally, note that if x and y are positive (as in the drawing) then a_x and a_y (both of which point to the sun in the center) are negative. Putting it all together…

$$a_x = -GM\frac{x}{(x^2 + y^2)^{3/2}}$$
$$a_y = -GM\frac{y}{(x^2 + y^2)^{3/2}}$$

Final Answer: $a_x = -GMx/(x^2 + y^2)^{3/2}$, $a_y = -GMy/(x^2 + y^2)^{3/2}$

(c) Use the acceleration components you just wrote to write two coupled differential equations for $x(t)$ and $y(t)$.
Final Answer: $d^2x/dt^2 = -GMx/(x^2 + y^2)^{3/2}$, $d^2y/dt^2 = -GMy/(x^2 + y^2)^{3/2}$

(d) Verify that $x(t) = a\cos\left(\sqrt{GM/a^3}\, t\right)$, $y(t) = a\sin\left(\sqrt{GM/a^3}\, t\right)$ is a solution to the coupled equations you wrote for any value of a. What kind of motion does this solution represent?
Solution: Since $(x^2 + y^2)^{3/2}$ appears in both equations, let's start there.

$$x^2 + y^2 = a^2\cos^2\left(\sqrt{GM/a^3}\, t\right) + a^2\sin^2\left(\sqrt{GM/a^3}\, t\right) = a^2$$

$$(x^2 + y^2)^{3/2} = a^3$$

With that out of the way, let's look at the differential equation for x.

$$\frac{d^2x}{dt^2} = -\frac{GM}{a^2}\cos\left(\sqrt{GM/a^3}\,t\right)$$
$$-\frac{GMx}{(x^2+y^2)^{3/2}} = -\frac{GM}{a^2}\cos\left(\sqrt{GM/a^3}\,t\right)$$

The y equation works out similarly.
In general the parametric equations $x = a\cos(\theta)$, $y = a\sin(\theta)$ describe a circle of radius a. This solution matches that form. Since the angle θ is proportional to t the Earth is tracing out a circle around the sun. (The real Earth moves in an ellipse, but this shows that a circle would be a possible orbit.)
Final Answer: Circle

1.148 Walk-Through: Decoupling Equations. In this problem you will solve the equations $f'(t) = f(t) + g(t)$, $g'(t) = 3f(t) - g(t)$. We will refer to these as the f' equation and the g' equation respectively.

(a) Differentiate both sides of the f' equation to get f'' in terms of f' and g'.
Final Answer: $f'' = f' + g'$

(b) Substitute g' from the g' equation into your answer to get f'' in terms of f, f', and g.
Final Answer: $f'' = f' + 3f - g$

(c) Solve for g in the original f' equation and plug this into your answer to get a decoupled, second- order differential equation for f.
Final Answer: $f'' = 4f$

(d) Find the general solution for $f(t)$ by inspection.
Final Answer: $f(t) = Ae^{2t} + Be^{-2t}$

(e) Plug this solution for f into the f' equation and solve (algebraically) for g.
Final Answer: $g = Ae^{2t} - 3Be^{-2t}$

(f) Plug your general solution for f and g into the original equations and verify that they work.
Solution:

$$\begin{aligned} f' \text{ equation: } & 2Ae^{2t} - 2Be^{-2t} \\ & = Ae^{2t} + Be^{-2t} + Ae^{2t} - 3Be^{-2t} \\ g' \text{ equation: } & 2Ae^{2t} + 6Be^{-2t} \\ & = 3Ae^{2t} + 3Be^{-2t} - Ae^{2t} + 3Be^{-2t} \end{aligned}$$

Both are true, so it works.

1.149 $f' = 3g, g' = 6f$

Solution: $f' = 3g$ becomes $f'' = 3g'$. Plugging this into the second equation makes it $f''/3 = 6f$ or $f'' = 18f$ with solution $f = Ae^{\sqrt{18}x} + Be^{\sqrt{18}x}$. You can plug this into either original equation to find $g(x)$.

$$f(x) = Ae^{\sqrt{18}x} + Be^{-\sqrt{18}x}$$
$$g(x) = \frac{A\sqrt{18}}{3}e^{\sqrt{18}x} - \frac{B\sqrt{18}}{3}e^{-\sqrt{18}x}$$

Final Answer: $f(x) = Ae^{\sqrt{18}x} + Be^{\sqrt{18}x}$, $g(x) = A\sqrt{18}e^{\sqrt{18}x}/3 - B\sqrt{18}e^{\sqrt{18}x}/3$

1.150 $f' = f + g, g' = f$

Solution: Use $g' = f$ and its derivative $g'' = f'$ to replace both f and f' in the first equation and you get $g'' = g' + g$ or $g'' - g' - g = 0$. Guessing a solution of the form $g = e^{kx}$ leads to $k^2 - k - 1 = 0$. So $g = Ae^{k_1 x} + Be^{k_2 x}$ with k_1 and k_2 given by the above quadratic equation, and f is the derivative of that.

$$f(x) = Ak_1 e^{k_1 x} + Bk_2 e^{k_2 x}$$
$$g(x) = Ae^{k_1 x} + Be^{k_2 x}$$
$$k_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Final Answer: $f(x) = Ak_1 e^{k_1 x} + Bk_2 e^{k_2 x}$, $g(x) = Ae^{k_1 x} + Be^{k_2 x}$ where $k_{1,2} = \left(-1 \pm \sqrt{5}\right)/2$.

1.151 $f' = af + bg, g' = cf + dg$

Solution: We will use the first equation and its derivative to solve for both g and g'.

$$f' = af + bg \quad \rightarrow \quad g = \frac{f' - af}{b}$$
$$f'' = af' + bg' \quad \rightarrow \quad g' = \frac{f'' - af'}{b}$$

Now plug both of these into the second equation.

$$\frac{f'' - af'}{b} = cf + d\frac{f' - af}{b}$$

Simplify and put into standard form.

$$f'' - (a + d)f' + (ad - bc)f = 0$$

Assume a solution of the form $f = e^{kx}$ and the solution is $f = Ae^{k_1 x} + Be^{k_2 x}$ where k_1 and k_2 are given by the quadratic equation $k^2 - (a + d)k + (ad - bc) = 0$.

We already know that $g = (f' - af)/b$ so we can plug our known f into that and get

$$g = \frac{1}{b}\left(Ak_1 e^{k_1 x} + Bk_2 e^{k_2 x} - aAe^{k_1 x} - aBe^{k_2 x}\right)$$

So now we have our solutions.

$$f(x) = Ae^{k_1 x} + Be^{k_2 x}$$
$$g(x) = \frac{A}{b}(k_1 - a)e^{k_1 x} + \frac{B}{b}(k_2 - a)e^{k_2 x}$$
$$k_{1,2} = \frac{a + d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2}$$

Final Answer: $f(x) = Ae^{k_1 x} + Be^{k_2 x}$, $g(x) = (A/b)(k_1 - a)e^{k_1 x} + (B/b)(k_2 - a)e^{k_2 x}$ where $k_{1,2} = (1/2)\left(a + d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}\right)$.

1.152 $f' = af^2/g$, $g' = 2af + bg$. *Hint*: You should end up with a simple linear equation for f''.

Solution: We're going to solve the first equation to plug in for g in the second.

$$f'g = af^2 \quad \rightarrow \quad f''g + f'g' = 2aff' \quad \rightarrow \quad g' = \frac{2aff' - f''g}{f'} = 2af - \frac{f''}{f'}g$$

So we plug that in for g', and plug af^2/f' in for g, and the second equation becomes this.

$$2af - \frac{f''}{f'}\left(\frac{af^2}{f'}\right) = 2af + b\frac{af^2}{f'} \quad \rightarrow \quad -\frac{f''}{f'} = b \quad \rightarrow \quad f'' = -bf'$$

That is a second- order equation for f but we can think of it as a first- order equation for f'. Whatever f' is, when you take its derivative it multiplies by $-b$. In other words $f' = Ae^{-bx}$. When we integrate that to find f we can absorb the $-b$ into the arbitrary constant A, but we still have to add another arbitrary constant, so $f = Ae^{-bx} + B$.

Now remember that $g = af^2/f'$ which is $a(Ae^{-bx} + B)^2/(-Abe^{-bx})$. To reach the form below we squared out the top and then divided each term by the e^{-bx} on the bottom.

$$f(x) = Ae^{-bx} + B$$
$$g(x) = -\frac{a}{Ab}\left(A^2e^{-bx} + 2AB + B^2e^{bx}\right)$$

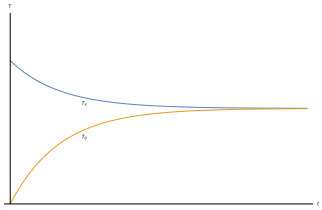
Final Answer: $f(x) = Ae^{-bx} + B$, $g(x) = -a(A^2e^{-bx} + 2AB + B^2e^{bx})/(Ab)$

1.153 Bars A and B are in contact with each other.

(a) Write a pair of coupled ODEs for T_A and T_B.
Final Answer: $T_A' = k_A(T_B - T_A)$, $T_B' = k_B(T_A - T_B)$

(b) Looking at your equations, describe what will happen to T_A and T_B over time. If they start with $T_A > T_B$ what will happen to the two temperatures initially? How will it change over time? What will happen at very late times?
Solution: The hotter one will cool down and the colder one will heat up—rapidly at first, then slower over time as their temperatures approach each other. They will approach the equilibrium state of having the same temperature.

(c) Based on your answers, draw a qualitative sketch showing T_A and T_B vs. time on the same plot. Your plot should show how the functions behave at early and late times. You do not need to include any numbers on your axes.
Solution:



(d) Solve the differential equations you wrote by decoupling them. Verify that the solutions match the behavior you described qualitatively above.
Solution: Use the first equation to solve for T_B and T_B'.

$$T_A' = k_A(T_B - T_A) \quad \rightarrow \quad T_B = \frac{T_A'}{k_A} + T_A \quad \rightarrow \quad T_B' = \frac{T_A''}{k_A} + T_A'$$

Now plug both into the second equation.

$$\frac{T_A''}{k_A} + T_A' = k_B\left(T_A - \frac{T_A'}{k_A} - T_A\right)$$

This simplifies to $T_A'' = -(k_A + k_B)T_A'$. That's a second- order differential equation for T_A but we can look at it as a first- order differential equation for T_A'; when you take its derivative, it multiplies by $-(k_A + k_B)$. So $T_A' = Ce^{-(k_A+k_B)t}$. When we integrate that to find T_A we can absorb the $-(k_A + k_B)$ into the arbitrary constant, but we have to add another arbitrary constant, so $T_A = C_1 e^{-2(k_A+k_B)t} + C_2$.
We already have a formula for T_B in terms of T_A.

$$T_B = \frac{T_A'}{k_A} + T_A = \frac{-2C_1(k_A + k_B)e^{-2(k_A+k_B)t}}{k_A} + C_1 e^{-2(k_A+k_B)t} + C_2$$

By splitting up the $(k_A + k_B)$ in the numerator of the fraction you can simplify this and end up in the following form.

$$T_A = C_1 e^{-2(k_A+k_B)t} + C_2$$
$$T_B = -C_1\left(2k_B/k_A + 1\right)e^{-2(k_A+k_B)t} + C_2$$

We can see that as the decaying exponentials approach zero, both temperatures will approach a common value (C_2). We can also see that if $C_1 > 0$ then T_A starts out higher

than C_2 and goes down, while T_B starts out lower than C_2 and goes up, and vice- versa if $C_2 < 0$.

Final Answer: $T_A = C_1 e^{-2(k_A+k_B)t} + C_2$, $T_B = -C_1 \left(2k_B/k_A + 1\right) e^{-2(k_A+k_B)t} + C_2$

1.154 Bars A, B, and C are stacked so that A and B are touching and B and C are touching, but not A and C.

(a) Write a set of coupled ODEs for T_A, T_B, and T_C.
Final Answer: $T_A' = k_A(T_B - T_A)$, $T_C' = k_C(T_B - T_C)$, and $T_B' = k_B(T_A + T_C - 2T_B)$

(b) What would have to be true of T_A, T_B, and T_C in order for none of them to change? You can use physical intuition to help guide you, but you must explain your answer in terms of the ODEs you wrote.
Solution: All three have to be the same. If $T_A \neq T_B$ then $T_A' \neq 0$, and similarly if $T_B \neq T_C$ then $T_C' \neq 0$. If all three are the same then all three derivatives come out zero.
Final Answer: All three have to be the same.

(c) If the bars start out with $T_A > T_B = T_C$, describe what will happen to the temperatures initially? What will happen a short time later? What will happen a very long time later?
Solution: Initially heat will flow from A to B, so T_A will lower while T_B rises. This will create a difference between bars B and C so heat will start to flow from B to C as well. (So heat has flowed from A to C, but indirectly.) For a while you will see $T_A > T_B > T_C$. Eventually they will all approach a common temperature lower than the initial T_A but higher than the initial T_B.

1.155 Bar A is touching bar B, which is in contact with a room at constant temperature T_R. (Bar A is insulated from the room, so it can only exchange heat with bar B. The room is large so it will affect bar B without being affected by it.)

(a) Write a pair of coupled ODEs for T_A and T_B.
Final Answer: $T_A' = k_A(T_B - T_A)$, $T_B' = k_B(T_A + T_R - 2T_B)$

(b) What do you expect to happen to T_A and T_B in the long term?
Solution: They will both approach T_R

(c) Solve the ODEs you wrote by decoupling them. Take the limit of your solution as $t \to \infty$ and verify that they match your predictions.
Solution:

$$T_A' = k_A(T_B - T_A) \quad \rightarrow \quad T_B = \frac{T_A'}{k_A} + T_A \quad \rightarrow \quad T_B' = \frac{T_A''}{k_A} + T_A'$$

Now plug both of those into the second equation.

$$\begin{aligned}
&\frac{T_A''}{k_A} + T_A' \\
&= k_B \left[T_A + T_R - 2\left(\frac{T_A'}{k_A} + T_A \right) \right] \\
&\frac{T_A''}{k_A} + T_A' \\
&= k_B \left[T_R - 2\frac{T_A'}{k_A} - T_A \right] \\
&T_A'' + k_A T_A' \\
&= k_A k_B T_R - 2k_B T_A' - k_A k_B T_A \\
&T_A'' + (k_A + 2k_B) T_A' + k_A k_B T_A \\
&= k_A k_B T_R
\end{aligned}$$

We can solve the complementary homogeneous equation by guessing a solution of the form $T_A = e^{pt}$. We end up solving a quadratic equation for p.

$$p = \frac{-(k_A + 2k_B) \pm \sqrt{(k_A + 2k_B)^2 - 4k_A k_B}}{2} = \frac{-(k_A + 2k_B) \pm \sqrt{k_A^2 + 4k_B^2}}{2}$$

Two things to note about that. First, because $k_A^2 + 4k_B^2$ is always positive, we always get two real solutions. Second, because $k_A + 2k_B \geq \sqrt{k_A^2 + 4k_B^2}$ both solutions are always negative. So we're going to get decaying exponential functions.
The very simple particular solution that suggests itself is $T_A = T_R$ since that constant solution makes $T_A' = T_A'' = 0$. So when the exponential solutions decay the solution will approach T_R as it should.
Final Answer: $T_A = T_R + C_1 e^{-p_1 t} + C_2 e^{-p_2 t}$ where $p_{1,2} = (1/2)\left(k_A + 2k_B \pm \sqrt{k_A^2 + 4k_B^2} \right)$

1.156 The Discovery Exercise (Section 1.7.1) presented a simplified model of a predator-prey relationship. A more commonly used model, although still too simple for some situations, is the "Lotka–Volterra equations," sometimes called the "predator-prey equations."

$$dR/dt = \alpha R - \beta RF$$
$$dF/dt = \gamma RF - \delta F$$

R and F represents the populations of rabbits and foxes, and the Greek letters represent positive constants.

(a) What would happen to the rabbit population if there were no foxes? You may use common sense to check your answer, but you must explain how you can figure out your answer from the Lotka–Volterra equations.
Solution: If there are no foxes, $dR/dt = \alpha R$ with solution $R = R_0 e^{\alpha t}$. Unchecked by predators, the rabbit population will grow exponentially: the more rabbits you have, the more you get.
Final Answer: Exponential growth

(b) Similarly, explain using these equations what would happen to the fox population if there were no rabbits.
Solution: If there are no foxes, $dF/dt = -\delta F$ with solution $F = F_0 e^{-\delta t}$. With no food source the fox population will die off quickly.
Final Answer: Exponential decay

(c) What values would R and F have to have in order for their populations to be unchanging? Give both answers to this, one of which you can think of just by looking at the equations. The other one will require some calculation.
Solution: The "just looking" solution is $R = F = 0$. In that case $dR/dt = dF/dt = 0$. If there are no rabbits or foxes, nothing ever happens.
The "some calculations" solution starts by translating "populations are unchanging" to "derivatives are zero." This gives us two simultaneous equations to solve for R and F.

$$\alpha R - \beta RF = 0$$
$$\gamma RF - \delta F = 0$$

If $R \neq 0$ then we can divide the top equation by R and then solve to find $F = \alpha/\beta$, and similarly the bottom equation becomes $R = \delta/\gamma$.
Final Answer: $R = F = 0$ or $R = \delta/\gamma$, $F = \alpha/\beta$

(d) The product RF appears in both equations, adding to the number of foxes and subtracting from the rabbits. Why?
Solution: RF quantifies the probability of a fox encountering a rabbit. Such an encounter is good for the fox, not so good for the rabbit.

(e) The simplest population growth for the foxes would be $dF/dt = \delta F$: "the more foxes you have, the more you get." But in the Lotka–Volterra equations, δF is subtracted from dF/dt: "the more foxes you have, the fewer you get." Why? (*Hint*: You cannot answer this question without thinking about the whole scenario.)
Solution: The foxes are sharing a finite supply of food (rabbits). If you have ten rabbits, one fox can live like a king, but twenty foxes are going to be hungry.

1.157 *[This problem depends on Problem 156.]* Decouple the Lotka–Volterra equations to get a differential equation for $R(t)$. Why is this technique not particularly useful for this case?

Solution: Solve the first equation for F and F'.

$$F = \frac{\alpha R - dR/dt}{\beta R} = \frac{1}{\beta}\left(\alpha - \frac{dR/dt}{R}\right)$$

$$\frac{dF}{dt} = \frac{1}{\beta}\left(-\frac{R\left(d^2R/dt^2\right) - (dR/dt)^2}{R^2}\right)$$

Rewrite the second equation as $dF/dt = (\gamma R - \delta)F$ and plug in the values from above.

$$-\frac{1}{\beta}\left(\frac{R\left(d^2R/dt^2\right) - (dR/dt)^2}{R^2}\right) = (\gamma R - \delta)\frac{1}{\beta}\left(\alpha - \frac{dR/dt}{R}\right)$$

$$R\frac{d^2R}{dt^2} - \left(\frac{dR}{dt}\right)^2 - \left(\gamma R^2 - \delta R\right)\frac{dR}{dt} + \alpha\gamma R^3 - \alpha\delta R^2 = 0$$

Because this is a non-linear second- order equation, it will be very difficult if not impossible to solve.

Final Answer: $R\left(d^2R/dt^2\right) - (dR/dt)^2 - \left(\gamma R^2 - \delta R\right)(dR/dt) + \alpha\gamma R^3 - \alpha\delta R^2 = 0$

1.158 Consider the Romeo and Juliet problem with proper units. Here R and J are Romeo's and Juliet's love for each other and α and β are positive constants.

$$dR/dt = \alpha J$$
$$dJ/dt = -\beta R$$

(a) If R and J are each measured in love-units, what are the units of α and β?
Final Answer: one over time

(b) Solve for $R(t)$ and $J(t)$ by decoupling the equations. Your final answer should have two arbitrary constants in it.
Solution:

$$\begin{aligned} J &= \frac{1}{\alpha}\frac{dR}{dt} \\ \frac{dJ}{dt} &= \frac{1}{\alpha}\frac{d^2R}{dt^2} \\ \frac{1}{\alpha}\frac{d^2R}{dt^2} &= -\beta R \\ \frac{d^2R}{dt^2} &= -\alpha\beta R \\ R &= A\sin\left(\sqrt{\alpha\beta}\,t\right) \\ &\quad + B\cos\left(\sqrt{\alpha\beta}\,t\right) \\ J &= \sqrt{\frac{\beta}{\alpha}}\left[A\cos\left(\sqrt{\alpha\beta}\,t\right)\right. \\ &\quad \left. -B\sin\left(\sqrt{\alpha\beta}\,t\right)\right] \end{aligned}$$

Final Answer: $R(t) = A\sin(\sqrt{\alpha\beta}\,t) + B\cos(\sqrt{\alpha\beta}\,t), J(t) = \sqrt{\beta/\alpha}\left[A\cos(\sqrt{\alpha\beta}\,t) - B\sin(\sqrt{\alpha\beta}\,t)\right]$

(c) Using the units you found for α and β, what units must your arbitrary constants have in order for your answer to make sense? Make sure that all terms are added to things with equivalent units and that all arguments of trig functions are unitless.
Solution: Since α and β have the same units the square root in the front of $J(t)$ is unitless, so both R and J have units of the arbitrary constants, which means A and B must both have love-units. The arguments of the sines and cosines are all $\sqrt{\alpha\beta}\,t$, which is unitless since α and β have units of inverse time.

(d) Suppose α is very large compared to β. What does that tell us about Romeo and Juliet? (Answer based on the original differential equations.) What effect will it have on their behavior? (Answer based on your solution.)
Solution: Looking at the differential equations $\alpha \gg \beta$ means Romeo's feelings are much more strongly affected by Juliet's than the reverse. Looking at the solutions, $\alpha \gg \beta$ means that although their feelings with oscillate with the same period as each other, the amplitude of Romeo's oscillations will be much greater. This makes sense; since his feelings are strongly affected by Juliet's he will swing wildly from passionate love to unadulterated loathing, while she moves back and forth between mild attraction and dislike.

1.172 *[This problem does not require a computer.]* In the example on Page 14 we solved the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 9)y = 0 \; (x \geq 0)$$

with the boundary condition $y(0) = 0$. Explain why we could have arrived at the same solution if all we had specified was that $y(0)$ must be finite. Using similar logic, explain why you *cannot* solve this equation with the boundary condition $y(0) = 1$.

Solution: Since $Y_3(0)$ is undefined we knew C_2 had to equal zero. That left us with $C_1 J_e(x)$, which is automatically equal to zero at $x = 0$ regardless of C_1. So the requirement that $y(0)$ exists and is finite is equivalent to requiring that it equal 0 in this case.

1.177 Exploration: Euler's Method By Hand *[This problem does not require a computer.]*

The simplest technique of numerically solving differential equations is "Euler's method." In this problem, you will use Euler's Method to answer the question: If $dy/dx = y$ and $y(0) = 1$, what is $y(1)$?

This problem asks us to follow the function from $x = 0$ to $x = 1$. We choose to take this journey in three steps, each of length $1/3$.

(a) You're going to draw a slope field for this equation. To start, draw the axes and draw a small segment of slope at the point $(0, 1)$. Explain how we know that the value of the slope at that point must be 1.
Solution: $y'(0) = y(0) = 1$
(See below for the picture.)

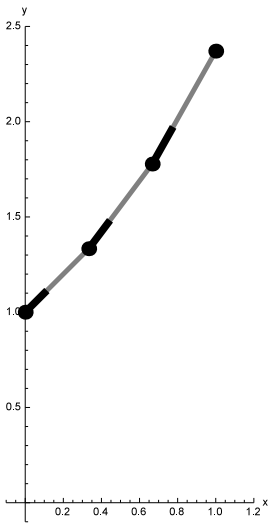
(b) Our next task is to find the value $y(1/3)$. Using the slope you found, explain how we can know that $y(1/3) \approx 4/3$.
Solution: From the definition of the derivative, $y'(x)$ is the amount y changes per unit change in x. That means $y(1/3) \approx y(0) + y'(0)(1/3 - 0) = 1 + 1(1/3) = 4/3$.

(c) Calculate the slope of the curve at $(1/3, 4/3)$ and use the result to draw the next entry in your slope field at that point.
Solution: $y'(1/3) = y(1/3) \approx 4/3$
(See below for the picture.)
Final Answer: $y'(1/3) = 4/3$

(d) Using the slope you just found, estimate the value of $y(2/3)$.
Solution: $y(2/3) \approx y(1/3) + y'(1/3)(2/3 - 1/3) = 4/3 + 4/3(1/3) = 16/9$
Final Answer: 16/9

(e) Calculate the slope at the new point you just found and use it to draw another entry on your slope field.
Solution: $y'(2/3) = y(2/3) \approx 16/9$
(See below for the picture.)
Final Answer: $y'(2/3) = 16/9$

(f) Use the last slope you calculated to estimate $y(1)$.
Solution: $y(1) \approx y(2/3) + y'(2/3)(1 - 2/3) = 16/9 + 16/9(1/3) = 64/27$
With this in place we can complete the slope field we've been drawing.



Final Answer: 64/27

(g) We began with the question "What is $y(1)$?" and you answered that question in Part f. Explain why your answer is not exactly correct. Your explanation should enable you to predict whether the actual $y(1)$ is higher or lower than your approximation.
Solution: For each point of the function $y(t)$ you calculated, you approximated the change in y with a constant slope based on the value of the slope at the previous point. However, in reality, the slope does not stay constant, but increases along the way. Therefore, in calculating each point of $y(t)$, you took the lowest value of the slope on each interval, and therefore, underestimated your value of $y(1)$.

(h) Solve the equation $dy/dx = y$ with initial condition $y(0) = 1$ analytically and find the exact, correct value for $y(1)$. If you didn't correctly predict whether your approximate answer would be too high or too low rethink your explanation and explain the result you did get.
Solution: By inspection, the general solution to $dy/dx = y$ is $y(x) = Ce^x$. The initial condition $y(0) = 1$ gives $C = 1$, so $y(x) = e^x$, and thus $y(1) = e \approx 2.7$. The approximate answer from Euler's method was $y(1) \approx 64/27 \approx 2.4$. As predicted, this is lower than the exact answer.
Final Answer: $y(1) = e$

1.179 Parts (a)–(e) below give five different scenarios for the mass of a spherical snowball $M(t)$. Match each one with the appropriate differential equation.

(a) Rolling down a hill, it gains 5 grams per /second.	I. $dM/dt = 2M$
(b) Every second, 5 grams melt.	II. $dM/dt = M - 5$
(c) Rolling, the mass doubles every second.	III. $dM/dt = -5$
(d) Rolling along, the mass triples every second.	IV. $dM/dt = 5$
(e) Every second the snowball picks up its own mass, but 5 grams melt.	V. $dM/dt = M$

Final Answer: (a) IV. (b) III. (c) V. (d) I. (e) II.

1.180 A thermostat is set to keep the house temperature at a constant 68°C. If the temperature rises too high, the air conditioner brings it down; if the temperature falls too low, the heater brings it up. Using u for the room temperature, which of the following differential equations best represents this situation? Assume every k is a positive constant with the appropriate units.

(a) $du/dt = 68k$
(b) $du/dt = 68kt$
(c) $du/dt = 68ku$
(d) $du/dt = k(u - 68)$
(e) $du/dt = k(68 - u)$
(f) $du/dt = k(u + 68)$
(g) $du/dt = 68k_1(u + k_2t)/(u - k_2t)$

Solution: The rate of change must depend on u, which eliminates a and b. For a given u the rate of change is the same regardless of the time, so t can't appear in the equation. That eliminates b and g. The rate of change is proportional to the difference between u and 68, so it must be d or e. We're told that k is positive and we know that if u is above 68 the temperature will come down (negative derivative), while if $u < 68$ it will come up $du/dt > 0$. That matches e but not d.

Final Answer: (e)

1.181 DE: $r'(\theta) + 3r(\theta) = \cos\theta$
Solution: $r = (\sin\theta + 3\cos\theta)/10 + Ce^{-3\theta}$
Condition: $r(0) = 1$

Solution:

$$r = \frac{1}{10}\sin\theta + \frac{3}{10}\cos\theta + Ce^{-3\theta}$$
$$r' = \frac{1}{10}\cos\theta - \frac{3}{10}\sin\theta - 3Ce^{3\theta}$$
$$r' + 3r = \cos\theta$$

So it works.

$$r(0) = \frac{3}{10} + C = 1$$

So $C = 7/10$.

Final Answer: $r = (\sin\theta + 3\cos\theta)/10 + (7/10)e^{-3\theta}$

1.182 DE: $dx/dt = (tx - x^2)/t^2$
Solution: $x = t/(\ln t + C)$
Condition: $x(1) = e$

Solution:

$$\begin{aligned} x &= \frac{t}{\ln t + C} \\ \frac{dx}{dt} &= \frac{\ln t + C - 1}{(\ln t + C)^2} \\ \frac{1}{t^2}(tx - x^2) &= \frac{1}{t^2}\left(\frac{t^2}{\ln t + C} - \frac{t^2}{(\ln t + C)^2}\right) \\ &= \frac{\ln t + C - 1}{(\ln t + C)^2} \end{aligned}$$

So it works.

$$x(1) = \frac{1}{C} = e$$

So $C = 1/e$.

Final Answer: $x = t/(\ln t + 1/e)$

1.183 DE: $\dfrac{dy}{dx} = \dfrac{(x+y)\ln(x+y) - (x+2)}{x+2}$

Solution: $y = e^{C(x+2)} - x$

Condition: $y(0) = 10$

Solution:

$$\begin{aligned} y &= e^{C(x+2)} - x \\ \frac{dy}{dx} &= Ce^{C(x+2)} - 1 \\ x + y &= e^{C(x+2)} \\ \frac{(x+y)\ln(x+y) - (x+2)}{x+2} &= \frac{C(x+2)e^{C(x+2)} - (x+2)}{x+2} \\ &= Ce^{C(x+2)} - 1 \end{aligned}$$

So it works.

$$y(0) = e^{2C} = 10$$

So $C = (1/2)\ln(10)$.

Final Answer: $y = 10^{x/2+1} - x$

1.184 DE: $\dfrac{dy}{dx} = \dfrac{(\ln x - 1)(y^2 + 1)}{xy \ln x}$

Solution: $y = \sqrt{\dfrac{Cx^2}{(\ln x)^2} - 1}$

Condition: $y(e^2) = 3$

Solution:

$$y = \sqrt{\frac{Cx^2}{(\ln x)^2} - 1}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\left(\frac{Cx^2}{(\ln x)^2} - 1\right)^{-1/2}\left(\frac{2Cx(\ln x)^2 - 2Cx(\ln x)}{(\ln x)^4}\right) \\
&= \left(\frac{Cx^2}{(\ln x)^2} - 1\right)^{-1/2}\left(\frac{Cx \ln x - Cx}{(\ln x)^3}\right) \\
&= \frac{Cx(\ln x - 1)}{y(\ln x)^3}\,\frac{(\ln x - 1)(y^2 + 1)}{xy \ln x} \\
&= \frac{(\ln x - 1)Cx^2/(\ln x)^2}{xy \ln x} \\
&= \frac{Cx(\ln x - 1)}{y(\ln x)^3}
\end{aligned}$$

So it works.

$$y(e^2) = \sqrt{\frac{Ce^2}{4} - 1} = 3$$

So $C = 40/e^2$.

Final Answer: $y = \sqrt{\dfrac{40x^2}{e^2(\ln x)^2} - 1}$

1.185 DE: $V''(x) - 4V(x) = 6x - 4x^3$
Solution: $V(x) = x^3 + C_1 e^{2x} + C_2 e^{-2x}$
Conditions: $V(0) = 0$, $V(\ln 2) = 1$

Solution:

$$\begin{aligned} V(x) &= x^3 + C_1 e^{2x} + C_2 e^{-2x} \\ V'(x) &= 3x^2 + 2C_1 e^{2x} - 2C_2 e^{-2x} \\ V''(x) &= 6x + 4C_1 e^{2x} + 4C_2 e^{-2x} \\ V''(x) - 4V(x) &= 6x - 4x^3 \end{aligned}$$

So it works.

$$\begin{aligned} V(0) &= C_1 + C_2 = 0 \\ V(\ln 2) &= (\ln 2)^3 + 4C_1 + C_2/4 = 1 \end{aligned}$$

Solving simultaneously, $C_1 = -(4/15)\left((\ln 2)^3 - 1\right)$ and $C_2 = (4/15)\left((\ln 2)^3 - 1\right)$.

Final Answer: $V = x^3 + (4/15)\left((\ln 2)^3 - 1\right)\left(e^{-2x} - e^{2x}\right)$

1.186 DE: $s''(t) - 9s + 8 = \sin t$
Solution: $s(t) = 8/9 - (\sin t)/10 + C_1 e^{3t} + C_2 e^{-3t}$
Conditions: $s(0) = 0$, $s'(0) = 0$

Solution:

$$s(t) = \frac{8}{9} - \frac{\sin t}{10} + C_1 e^{3t} + C_2 e^{-3t}$$
$$s'(t) = -\frac{\cos t}{10} + 3C_1 e^{3t} - 3C_2 e^{-3t}$$
$$s''(t) = \frac{\sin t}{10} + 9C_1 e^{3t} + 9C_2 e^{-3t}$$
$$s''(t) - 9s + 8 = \sin t$$

So it works.

$$s(0) = \frac{8}{9} + C_1 + C_2 = 0$$
$$s'(0) = -\frac{1}{10} + 3C_1 - 3C_2 = 0$$

Solving simultaneously, $C_1 = -77/180$ and $C_2 = -83/180$.

Final Answer: $s = 8/9 - (\sin t)/10 - (77/180)e^{3t} - (83/180)e^{-3t}$

1.187 $dy/dt = ky$, $y(0) = p$

Solution: We can solve this by separating variables, but we can also solve it by thinking about what it says: "The derivative of $y(t)$ is the same as the original function $y(t)$ multiplied by a constant k." The solution is $y = Ae^{kt}$ for any constant A. Plugging in the initial condition, $y(0) = A = p$.

Final Answer: $y_{general}(t) = Ae^{kt}$, $y_{specific}(t) = pe^{kt}$

1.188 $d^2y/dt^2 = -16y, y(0) = 0, y'(0) = 2$

Solution: This is a simple harmonic oscillator equation with solution

$$y(t) = A\sin(4t) + B\cos(4t)$$

The first condition gives $B = 0$, so $y(t) = A\sin(4t)$. To plug in the second condition you need the derivative $y'(t) = 4A\cos(4t)$. Plugging in $y'(0) = 2$ gives $4A = 2$, so

$$y(t) = \frac{1}{2}\sin(4t)$$

Checking: Looking at this I can see that $y(0) = 0$ and $y'(t) = 2\cos(4t)$, so $y'(0) = 2$. To check the differential equation I calculate $y''(t) = -8\sin(4t)$. This does equal -16 times the original function, so the solution works.

Final Answer: $y_{general}(t) = A\sin(4t) + B\cos(4t)$, $y_{specific}(t) = (1/2)\sin(4t)$

1.189 $du/dx = u + 3$, $u(0) = 0$

Solution: A more standard form would be:

$$\frac{du}{dx} - u = 3$$

The complementary homogeneous equation is $du/dx - u = 0$ or $du/dx = u$. The function $u_c(x)$ is its own derivative, so $u_c(x) = Ae^x$.

To that we need to add a particular solution, and $u_p(x) = -3$ suggests itself because du_p/dx would then be zero.

The general solution is the sum of the complementary and particular solutions: $u(x) = Ae^x - 3$. Plugging in the initial condition, $u(0) = A - 3 = 0$ so $A = 3$.

Final Answer: $u_{general}(x) = Ae^x - 3$, $u_{specific}(t) = 3e^x - 3$

1.190 $du/dx = (2x + 1)u$, $u(0) = 1$

Solution: We cannot use linear superposition here because the equation is not linear. But we can separate the variables.

$$\int \frac{du}{u} = \int (2x + 1)dx$$
$$\ln u = x^2 + x + C$$
$$u = e^{x^2+x+C} = e^{x^2+x}e^C = Ae^{x^2+x}$$

Plugging in the initial condition, $u(0) = A = 1$.

Final Answer: $u_{general}(x) = Ae^{x^2+x}$, $u_{specific}(x) = e^{x^2+x}$

1.191 $d^2f/dx^2 - 5df/dx + 6f = 0, f(0) = 0, f(1) = 1$

Solution: Guess an exponential $f = e^{kx}$ and plug it in to get

$$k^2e^{kx} - 5ke^{kx} + 6e^{kx} = 0$$

Factoring out e^{kx} and factoring the resulting equation gives $k = 2$ or $k = 3$, so $f_{general}(x) = Ae^{2x} + Be^{3x}$. The condition $f(0) = 0$ gives $B = -A$ and the condition $f(1) = 0$ gives $A = 1/(e^2 - e^3)$

Final Answer: $f_{general}(x) = Ae^{2x} + Be^{3x}, f_{specific}(x) = (e^{2x} - e^{3x})/(e^2 - e^3)$

1.192 $d^2f/dx^2 - 3df/dx + 2f = 6e^{3x}, f(0) = 0, f'(0) = 0$

Solution: To get the complementary solution guess an exponential $f = e^{kx}$ and plug it in to get

$$k^2e^{kx} - 3ke^{kx} + 2e^{kx} = 0$$

Factoring out e^{kx} and factoring the resulting equation gives $k = 1$ or $k = 2$, so $f_C(x) = Ae^x + Be^{2x}$. To find a particular solution try $f = ae^{3x}$, which gives

$$9ae^{2x} - 9ae^{2x} + 2ae^{2x} = 6e^{2x}$$

That means $a = 3$ so $f_{general}(x) = Ae^x + Be^{2x} + 3e^{3x}$ The condition $f(0) = 0$ gives $B = -3 - A$, and the condition $f'(0) = 0$ gives

$$A + 2(-3 - A) + 9 = 0$$

So $A = 3$.

Final Answer: $f_{general}(x) = Ae^x + Be^{2x} + 3e^{3x}, f_{specific}(x) = 3e^x - 6e^{2x} + 3e^{3x}$

1.193 With sufficient food and no predators, the population of tribbles multiplies by 121 every day. However, the population is kept in check by a ravenous horde of vermicious knids, who eat 12,000 tribbles a day.

(a) Write a differential equation for the number of tribbles as a function of time.
Final Answer: $dP/dt = 120P - 12000$

(b) What is the equilibrium solution to your equation? Is it a stable or unstable equilibrium? How can you tell?
Solution: dP/dt is 0 when $P = 100$. Therefore, whenever the population reaches that number, it will always stay that number.
If $P > 100$, the derivative of P will be positive and will increase as P increases. If $P < 100$, the derivative of P will be negative and will decrease as P decreases. Therefore, if the population starts at any number other than 100 it will move away from that number. So, the point $P = 100$ is an unstable equilibrium.
Final Answer: $P = 100$; unstable

(c) Solve your differential equation with the initial condition $P(0) = 125$.
Solution: Using separation of variables or guess-and-check, you can determine the general solution to be $P = Ae^{120t} + 100$. Then, using the initial condition, you can determine the specific solution to be $P = 25e^{120t} + 100$.
Final Answer: $P = 25e^{120t} + 100$

1.194 Solve the differential equation

$$\frac{d^2y}{dt^2} = 3y$$

with initial conditions $y(0) = 0, y'(0) = 4\sqrt{3}$.

Solution: Rewrite the equation as $d^2y/dt^2 - 3y = 0$. Guess an exponential $y = e^{pt}$ and plug it in to get

$$p^2e^{pt} - 3e^{pt} = 0$$

Factoring out e^{pt} and solving the resulting equation gives $p = \pm\sqrt{3}$, so $y_{general}(t) = Ae^{\sqrt{3}t} + Be^{-\sqrt{3}t}$.

The condition $y(0) = 0$ gives $A = -B$. To plug in the second condition, you need the derivative.

$$y'(t) = \sqrt{3}\left(Ae^{\sqrt{3}t} - Be^{-\sqrt{3}t}\right) = A\sqrt{3}\left(e^{\sqrt{3}t} + e^{-\sqrt{3}t}\right)$$

So $y'(0) = 2A\sqrt{3} = 4\sqrt{3}$ so $A = 2$ and $B = -2$.

Final Answer: $y = 2e^{\sqrt{3}t} - 2e^{-\sqrt{3}t}$

1.195 $dy/dx = e^{x+y}$.

(a) Solve.

Solution: We can separate the variables if we start by rewriting the problem a bit.

$$\begin{aligned}
\frac{dy}{dx} &= e^x e^y \\
\int e^{-y} dy &= \int e^x dx \\
-e^{-y} &= e^x + C \\
e^{-y} &= -e^x + C \\
-y &= \ln(C - e^x) \\
y &= -\ln(C - e^x)
\end{aligned}$$

Final Answer: $y(x) = -\ln(C - e^x)$

(b) Find the solution that contains the point $y(0) = -2$.

Solution: $y(0) = -\ln(C-1) = -2$ so $\ln(C-1) = 2$ so $C - 1 = e^2$ so $C = e^2 + 1$.

Final Answer: $y(x) = -\ln(e^2 - e^x + 1)$

(c) Demonstrate that your solution satisfies both the differential equation and the initial condition.

Solution:

$$\begin{aligned}
y(x) &= -\ln(e^2 - e^x + 1) \\
\frac{dy}{dx} &= \frac{e^x}{e^2 - e^x + 1} \\
e^y &= \frac{1}{e^2 - e^x + 1} \\
e^x e^y &= \frac{e^x}{e^2 - e^x + 1}
\end{aligned}$$

So it solves the differential equation.

$$y(0) = -\ln(e^2 - 1 + 1) = -2$$

So it solves the initial condition.

1.196 The function $y = f(x)$ is the solution to the differential equation $dy/dx = (y-2)^2/(x-2)$ that goes through the point $(3, 0)$.

(a) Find the slope of this curve at the point $(3, 0)$.
Solution: Plugging the point into our equation for slope, $dy/dx = (-2)^2/1 = 4$.
Final Answer: 4

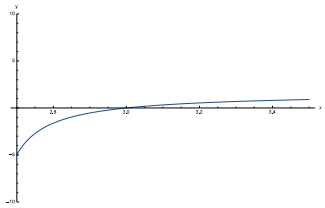
(b) Find a formula for the *concavity* of this curve as a function of x and y. (Remember that concavity is the derivative with respect to x of dy/dx. After you use the quotient rule, your formula for concavity will have x, y, and dy/dx in it. Then you can replace dy/dx with $(y-2)^2/(x-2)$ to get a function of x and y.)
Solution:

$$\frac{d^2y}{dx^2} = \frac{2(x-2)(y-2)(dy/dx) - (y-2)^2}{(x-2)^2} = \frac{2(y-2)^3 - (y-2)^2}{(x-2)^2}$$
$$= (y-2)^2\frac{2(y-2)-1}{(x-2)^2} = \frac{(2y-5)(y-2)^2}{(x-2)^2}$$

Final Answer: $d^2y/dx^2 = (2y-5)(y-2)^2/(x-2)^2$

(c) Find the concavity of this curve at the point $(3, 0)$.
Final Answer: -20

(d) Based on your answers to Parts (a) and (c), draw a quick sketch of the curve around the point $(3, 0)$.
Solution:



(e) Find the function $y = f(x)$.
Solution: We begin by finding the general solution by separation of variables.

$$\int \frac{dy}{(y-2)^2} = \int \frac{dx}{x-2}$$
$$-\frac{1}{y-2} = \ln(x-2) + C$$
$$y - 2 = -\frac{1}{\ln(x-2)+C}$$
$$y = -\frac{1}{\ln(x-2)+C} + 2$$

Our initial condition tells us that $y(3) = 0$.

$$-\frac{1}{C} + 2 = 0$$

Solving, $C = 1/2$.

$$y = 2 - \frac{2}{2\ln(x-2)+1}$$

Final Answer: $y = 2 - 2/[2\ln(x-2)+1]$

1.197 The function $V(t)$ follows the differential equation $dV/dt = 2V^2 + V^3 - V^4$. Predict the long-term behavior of V. Your answer will consist of several different statements of the form "If V starts in this range, then it will head toward…"

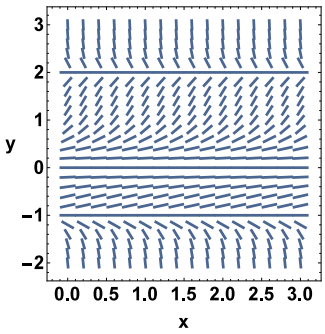
Solution: Factoring the equation, you get:

$$dV/dt = -V^2(V-2)(V+1)$$

The equilibrium solutions are $V = -1, 0, 2$.

- If V starts at any of these three values, it will stay at that value forever.
- If the system starts with $V < -1$ then dV/dt will be negative and decreasing with decreasing V values, so $V(t)$ will head toward negative infinity.
- If the system starts with $-1 < V < 0$ then dV/dt will be positive, so $V(t)$ will increase and approach $V = 0$ from below.
- If the system starts with $0 < V < 2$ then dV/dt will be positive, but this time $V(t)$ will increase and approach $V = 2$ from below.
- If the system starts with $V > 2$ then dV/dt will be negative, so $V(t)$ will decrease and approach $V = 2$ from above.

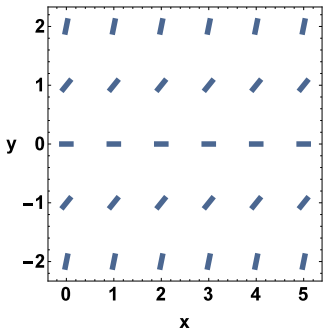
These conclusions are all confirmed if we make a slope field for the equation.



1.198 Consider the differential equation $dy/dt = y^2$.

(a) Begin by drawing a slope field at all integer points in $t \in [0, 5]$, $y \in [-2, 2]$.

Solution:



(b) Based on looking at your slope field, what is $\lim_{t\to\infty} y(t)$ if $y(0) = 1$? If $y(0) = -1$?

Solution: For $y(0) = 1$ the slope field shows it increasing faster and faster, so it will approach ∞.

For $y(0) = -1$ it is also increasing, but more slowly as it approaches the t-axis, so it will asymptotically approach zero.

Final Answer: $y(0) = 1$ approaches ∞, $y(0) = -1$ approaches $y = 0$.

(c) Now solve the differential equation using separation of variables. Your solution should have an arbitrary constant C.

Solution:

$$\int \frac{dy}{y^2} = \int dt$$
$$-\frac{1}{y} = t + C$$
$$y = -\frac{1}{t + C}$$

Final Answer: $y(t) = -1/(t + C)$

(d) Find two specific solutions for the two initial conditions $y(0) = 1$ and $y(0) = -1$.

Solution:

$$-\frac{1}{C} = 1 \quad \rightarrow \quad C = -1$$
$$-\frac{1}{C} = -1 \quad \rightarrow \quad C = 1$$

Final Answer: $y = -1/(t - 1)$, $y = -1/(t + 1)$

(e) Find $\lim_{t\to\infty} y(t)$ for your two specific functions.

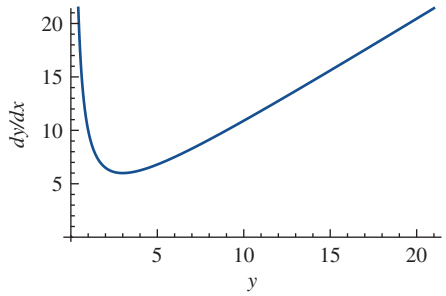
Final Answer: Both zero

(f) In one case, you should find that the limit you predicted based on your slope field does not match the limit you predicted based on your solution. Explain why one (or both) of your predictions was wrong.

Solution: For the starting condition $y(0) = -1$ the system acts exactly as predicted, gradually rising toward $y = 0$.

For the starting condition $y(0) = 1$ we predicted that y would rise toward infinity. And indeed it does—not as $t \to \infty$, but as $t \to 1^-$ it blows up. After that vertical asymptote it starts life again in negative y-values, rising up toward zero. This may not have been perfectly predictable from the slope field, but it is consistent with it.

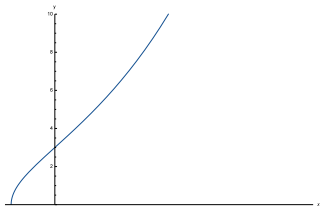
1.200 The drawing below shows dy/dx as a function of y. $\lim_{y\to 0^+}(dy/dx) = \infty$ and $\lim_{y\to\infty}(dy/dx) = \infty$, and the absolute minimum occurs where $y = 3$ and $dy/dx = 6$. The function is undefined for $y \le 0$.



Sketch the function $y(x)$ that goes through the point $x = 0, y = 3$.

Solution: From the given plot we know that dy/dx is always positive for positive y, so the function will increase. As y increases dy/dx will get larger so the slope will increase, meaning the curve will be concave up.

For $x < 0$ we also know y was increasing, but as y increased towards 3 the slope dy/dx was decreasing. As we go to more negative x we know dy/dx approaches infinity as $y \to 0$, so the slope becomes vertical.



Since the graph gives no information about $y < 0$ we can't continue to earlier values of x.

1.201 A dish contains an amount of bacteria $B(t)$. On average each bacteria cell produces one new offspring each day. At the same time an enzyme in the dish kills 1000 bacteria cells per day. Write a differential equation for $B(t)$ and solve it, assuming the dish starts with 1200 bacteria cells. (In writing your equation you should assume time is measured in days and B is measured in number of cells.) How many bacteria cells does the dish contain after ten days?

Solution: $dB/dt = B - 1000$

Using separation of variables, you get

$$\int \frac{1}{B-1000} dB = \int dt$$
$$\ln(B-1000) = t + C$$
$$B(t) = Ce^t + 1000$$

Now plug in the initial condition: $B(0) = C + 1000 = 1200$ so $C = 200$.

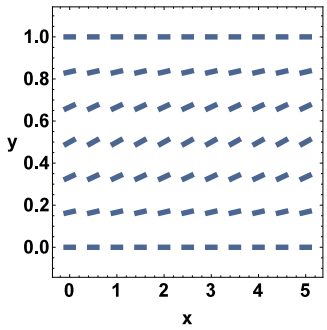
$$B(t) = 200e^t + 1000$$
$$B(10) = 200e^{10} + 1000 \approx 4,406,293$$

Final Answer: $B(10) = 200e^{10} + 1000$

1.202 In 1930, psychologist Louis L. Thurstone proposed[2] that learning follows the differential equation $dp/dt = k[p(1-p)]^{3/2}$ where p represents mastery of a task or topic on a scale of 0 to 1 and k is a positive constant.

(a) What are the equilibrium solutions, and what do they represent about learning?

Solution: This is easiest to see with a slope field.



$p = 0$ and $p = 1$ are the equilibrium solutions. They represent stages where no learning happens at all. The first equilibrium solution tells you that you can't learn anything if you don't have at least some knowledge of the topic to start with. The second equilibrium solution tells you that you can't learn any more if you already know all there is to know about the topic.

Final Answer: $p = 0$ and $p = 1$

(b) At what p-value does a student learn the fastest?

Solution: Since how much you learn is represented by dp/dt, you need to find where $k[p(1-p)]^{3/2}$ is a maximum. Take the derivative of this function and set it equal to 0.

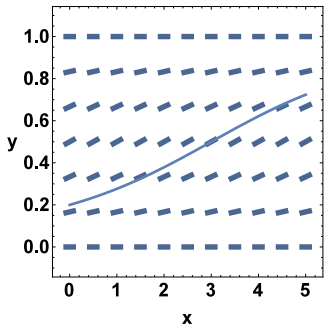
$$(3/2)k[p(1-p)]^{1/2}(1-2p) = 0$$

The critical points are $p = 0, 1, (1/2)$. Since we know from part (a) that $p = 0$ and $p = 1$ represent no learning at all, the point $p = (1/2)$ is where learning must be the maximum.

Final Answer: $p = 1/2$

(c) Sketch the general shape of a learning curve according to Thurstone's model.

Solution:



This equation can be solved analytically. We encourage you to have a computer solve it for you, which will give you a good appreciation for why sometimes an analytic solution to an ODE is not the best way to understand it.

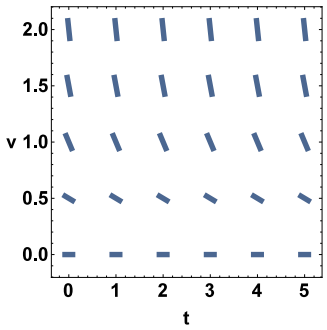
[2]Thurstone, L.L. The Learning Function. *Journal of General Psychology,* 1930, 3, 469-493.

1.203 We have worked several times with drag forces proportional to velocity. In some cases it is more accurate to model a drag force as proportional to velocity *squared*: $F = -kv^2$ where k is a positive constant.

(a) Explain why this law, as written, only makes sense for positive v-values.
Solution: A drag force should always have an opposite sign from the velocity. (In other words it should always slow you down, never speed you up!) Since this equation always gives a negative force, it can only be correct when v is positive.

(b) Using Newton's Second Law, write a first- order differential equation for the velocity of an object experiencing this drag force and no other forces.
Final Answer: $dv/dt = -kv^2/m$

(c) Draw a slope field at all integer points in $t \in [0, 5]$, $v \in [0, 2]$.
Solution:



(d) Based on looking at your slope field, what is $\lim_{t\to\infty} v(t)$ if $v(0) = 0$? If $v(0) = 1$?
Final Answer: The limit is $v = 0$ in both cases.

(e) Solve the differential equation using the initial condition $v(0) = v_0$.
Solution: Separating variables

$$\int \frac{dv}{v^2} = \int -\frac{k}{m}dt$$

$$-\frac{1}{v} = -\frac{k}{m}t + C$$

$$v = \frac{1}{(k/m)t - C}$$

Plugging in the initial condition gives $v_0 = -1/C$, so $C = -1/v_0$.
Final Answer: $v(t) = 1/\left[(k/m)t + (1/v_0)\right]$

(f) How long will it take the object to reach 1/10 of its original speed? Your answer will depend on k, m, and v_0.
Solution: Plugging in $v(t) = v_0/10$ gives

$$\frac{v_0}{10} = \frac{1}{(k/m)t + (1/v_0)}$$

$$\frac{k}{m}t + \frac{1}{v_0} = \frac{10}{v_0}$$

Final Answer: $t_{1/10} = 9m/(kv_0)$

(g) Find the body's *position* function $x(t)$, assuming $x(0) = 0$.

Solution:

$$x(t) = \int v(t)dt = \frac{m}{k} \ln\left(\frac{k}{m}t + \frac{1}{v_0}\right) + C$$

The initial condition leads to

$$C = -\frac{m}{k} \ln\left(\frac{1}{v_0}\right)$$

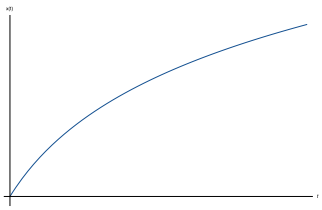
We can plug this in and simplify.

$$x(t) = \frac{m}{k} \ln\left(\frac{v_0 k}{m}t + 1\right)$$

Final Answer: $x(t) = (m/k)\ln(v_0 kt/m + 1)$

(h) Draw a quick sketch of $x(t)$ for $t \geq 0$.

Solution:



1.204 Black Holes and Hawking Radiation

In 1974 Stephen Hawking predicted that, due to quantum effects, black holes should lose mass. This process, sometimes called "evaporation," follows the equation $dM/dt = -\hbar c^4/(15360\pi G^2 M^2)$ where M is the mass of the black hole and all other quantities are constants.

(a) We can make this equation more tractable by grouping the constants together. Let $k = \hbar c^4/(15360\pi G^2)$ and rewrite the differential equation in a simpler form.
Final Answer: $dM/dt = -k/M^2$

(b) Looking at your differential equation, does it predict that black holes will grow or shrink? Does it predict that large black holes will change more slowly or more quickly, than small ones? Based on these facts, write a brief description of the life of a black hole.
Solution: Because dM/dt is negative, black holes will always shrink. For large M, dM/dt will be less negative, so large black holes will lose mass more slowly. For small M, dM/dt will be more negative, so small black holes will lose mass more quickly. Therefore, in the beginning of a black hole's life, its mass will be shrinking at a given rate, and as the black hole ages, it will continue shrinking faster and faster.
Final Answer: They will shrink faster and faster.

(c) Solve the differential equation with the initial condition $M = M_0$.
Solution: Using separation of variables, you get

$$\int M^2 dM = \int -k\, dt$$
$$\frac{1}{3}M^3 = -kt + C$$
$$M(t) = (C - 3kt)^{(1/3)}$$

Plugging in the initial condition $M(0) = M_0$ gives you $C = M_0^3$, so the specific solution is

$$M(t) = \left(M_0^3 - 3kt\right)^{(1/3)}$$

Final Answer: $M(t) = \left(M_0^3 - 3kt\right)^{(1/3)}$

(d) Look up values of $\hbar$ (Planck's constant divided by 2π), c (the speed of light), and G (the gravitational constant). Make sure they are all in standard SI units! Put them together to find the value of your constant k.
Final Answer: $k = 4 \times 10^{15}$ kg^3/s

(e) The most commonly observed type of black hole is a "stellar black hole," which has a mass comparable to that of the sun: 2×10^{31} kg. How long would such a black hole take to evaporate entirely?
Final Answer:

$$\left(M_0^3 - 3kt\right)^{(1/3)} = 0$$
$$M_0^3 - 3kt = 0$$
$$t = \frac{M_0^3}{3k}$$

Plugging in the given mass and our calculated k leads to approximately $t = 6.7 \times 10^{77}$ seconds, or 2.1×10^{70} years.
Final Answer: $t = 7 \times 10^{77}$ s

(f) The universe is roughly 1.4×10^{10} years old. Express the lifetime of a stellar black hole as a multiple of the current age of the universe.
Final Answer: The lifetime is about 10^{61} times the current age of the universe.

(g) It's possible that particle physics experiments will produce microscopic black holes with mass comparable to a proton: about 2×10^{-27} kg. As of this writing, there has been no evidence of such black holes being produced, but some people worry that if they were they would destroy the Earth. How long would such a black hole last? Is this something we should be worried about?[3]
Final Answer: $t = 7 \times 10^{-97}$ s

[3]Aside from the short lifetime of such black holes, there's a simpler argument for why they couldn't destroy Earth. Cosmic rays constantly strike the upper atmosphere with energies far greater than we can produce in a lab. If these experiments could produce anything that would destroy Earth it would have happened billions of years ago.

1.205 Rocket Science

A rocket engine produces thrust by burning and expelling fuel. Newton's Second Law $F = ma$ holds as always, but m is not a constant. Assume the rocket begins with a total mass of m_0 and burns off mass at a steady rate of k (measured in mass per unit time), producing a constant force of F.

(a) Write a function for the rocket's mass $m(t)$. (This is a high school algebra problem; don't try to make it complicated.)
Final Answer: $m(t) = m_0 - kt$

(b) Write a second- order differential equation for the rocket's position $x(t)$.
Solution: Using Newton's Second Law, $m(d^2x/dt^2) = F$, and the equation for m you got from part (a), you have

$$\frac{d^2x}{dt^2} = \frac{F}{m_0 - kt}$$

Final Answer: $d^2x/dt^2 = F/(m_0 - kt)$

(c) Because your equation gives d^2x/dt^2 as a function of t only (no x-dependence), you can integrate twice to find the general solution. Your final answer should have two independent arbitrary constants.
Solution: You can take the integrals using the substitution $u = m_0 - kt$:

$$\begin{aligned}
\frac{dx}{dt} &= \int \frac{F}{m_0 - kt}dt \\
&= -\frac{F}{k}\ln(m_0 - kt) + C_1 \\
x &= \int \left(-\frac{F}{k}\ln(m_0 - kt) + C_1\right)dt \\
&= \frac{F}{k^2}\left[(m_0 - kt)\ln(m_0 - kt) - (m_0 - kt)\right] + C_1 t + C_2 \\
&= \frac{F}{k^2}(m_0 - kt)\ln(m_0 - kt) + \left(C_1 + \frac{F}{k}\right)t + \left(C_2 - \frac{Fm_0}{k^2}\right)
\end{aligned}$$

Since $\left(C_1 + F/k\right)$ and $\left(C_2 - Fm_0/k^2\right)$ are just a fancy way of writing arbitrary constants, you can reduce them to simply C_1 and C_2, respectively, to get the final solution:

$$x(t) = \frac{F}{k^2}(m_0 - kt)\ln(m_0 - kt) + C_1 t + C_2$$

Final Answer: $x(t) = (F/k^2)(m_0 - kt)\ln(m_0 - kt) + C_1 t + C_2$

(d) Assuming the rocket begins its journey at rest at $x = 0$, solve for your arbitrary constants and find the function $x(t)$. Your answer will, of course, contain the constants F, m_0, and k.
Solution: Plugging in the first initial condition $x(0) = 0$, you get $C_2 = -(Fm_0/k^2)\ln m_0$. The fact that it starts at rest means $dx/dt(t = 0) = 0$, so we need to differentiate $x(t)$.

$$\frac{dx}{dt} = -\frac{F}{k}\ln(m_0 - kt) - \frac{F}{k} + C_1$$

Setting $(dx/dt)(0) = 0$ gives $C_1 = (F/k)(1 + \ln m_0)$. (You might think of just using the solution you found for dx/dt above, but you can't do that if you've redefined your

arbitrary constants since then.) Putting it all together gives you:

$$x(t) = \frac{F}{k^2}(m_0 - kt)\ln(m_0 - kt) + \frac{F}{k}(1 + \ln m_0)t - \frac{Fm_0}{k^2}\ln m_0$$
$$= \left(\frac{Fm_0}{k^2} - \frac{F}{k}t\right)\ln\left(\frac{m_0 - kt}{m_0}\right) + \frac{F}{k}t$$

Final Answer: $x(t) = \left(\frac{Fm_0}{k^2} - \frac{F}{k}t\right)\ln\left(\frac{m_0 - kt}{m_0}\right) + \frac{F}{k}t$

(e) If the rocket begins with a mass of 5000 kg and burns mass at a rate of 60 kg/sec, exerting a constant force of 5,000,000 N, how fast is it going after 50 seconds? How far has it traveled?
Solution: With the value we found for C_1,

$$\frac{dx}{dt} = -\frac{F}{k}\ln\left(\frac{m_0 - kt}{m_0}\right)$$

At $t = 50$ s, this gives

$$\frac{dx}{dt} = -\frac{5\times10^6\text{N}}{60\text{ kg/s}}\ln\left(\frac{5000\text{ kg} - (60\text{ kg/s})(50\text{ s})}{5000\text{ kg}}\right) = 7.6\times10^4\text{m/s}$$

Meanwhile the solution $x(t)$ gives

$$x(t) = \left(\frac{(5\times10^6\text{N})(5000\text{ kg})}{(60\text{ kg/s})^2} - \frac{5\times10^6\text{N}}{60\text{ kg/s}}(50\text{ s})\right)\ln\left(\frac{5000\text{ kg} - (60\text{ kg/s})(50\text{ s})}{5000\text{ kg}}\right)$$
$$+ \frac{5\times10^6\text{N}}{60\text{ kg/s}}(50\text{ s}) = 1.6\times10^6\text{m}$$

Notice that the units work out correctly everywhere because a N is a kg m/s^2.
Final Answer: $v(50\text{ s}) = 7.6\times10^4$m/s, $x(50\text{ s}) = 1.6\times10^6$m.

1.206 The Barometric Formula

A region is filled with fluid (gas or liquid). This fluid is everywhere subjected to the downward force of gravity, balanced by the "pressure gradient" (lower altitudes are at higher pressure). This equilibrium is represented by the equation $dP/dz = -\rho g$, where P is the pressure, z is height, ρ is density, and $g = 9.8$ m/s^2 is the acceleration due to gravity. (Deriving this equation is a physical problem unrelated to what we want to convey in this chapter, so we're just giving it to you.)

(a) For a liquid, density generally stays pretty constant. Treating ρ as a constant, find $P(z)$. Your answer should of course include an arbitrary constant.

Solution: Since dP/dz is a constant, we can just integrate: $P = -\rho g z + C$. So pressure increases linearly as depth decreases.

Final Answer: $P = -\rho g z + C$

(b) For a gas, the ideal gas law gives the density as $\rho = mP/(RT)$, where m is the molar mass and R is the ideal gas constant. Assuming constant temperature, find $P(z)$. This result is known as the "barometric formula."

Solution:

$$\begin{aligned} \frac{dP}{dz} &= -\frac{mPg}{RT} \\ \int \frac{dP}{P} &= -\int \frac{mg}{RT} dz \\ \ln P &= -\frac{mg}{RT} z + C \\ P &= Ce^{-(mg/RT)z} \end{aligned}$$

Final Answer: $P = Ce^{-(mg/RT)z}$

1.207 *[This problem depends on Problem 206.]* In Problem 206(b) you derived the barometric formula for pressure variation with depth by assuming a constant temperature. For the lower portion of the Earth's atmosphere, known as the troposphere, a more realistic model is a linear variation of temperature with altitude: $T = T_0 - kz$. Calculate $P(z)$ under this assumption.

Solution:

$$\frac{dP}{dz} = -\frac{mPg}{R(T_0 - kz)}$$

$$\int \frac{dP}{P} = -\frac{mg}{R} \int \frac{dz}{T_0 - kz}$$

$$\ln P = \frac{mg}{Rk} \ln(T_0 - kz) + C$$

$$P = \left(e^{\ln(T_0 - kz)}\right)^{mg/(Rk)} e^C$$

$$P = C(T_0 - kz)^{mg/(Rk)}$$

Final Answer: $P = C(T_0 - kz)^{mg/(Rk)}$

1.208 Newton's Law of Heating and Cooling

A small object (such as a hot cup of coffee or a cold soda) at temperature Q is placed inside a room with ambient temperature Q_r. According to "Newton's Law of Heating and Cooling," the rate of change of the object's temperature is proportional to the *difference* in temperature between the object and the room.

(a) Newton's Law can be expressed as $dQ/dt = k(Q - Q_r)$ or as $dQ/dt = k(Q_r - Q)$. Both are mathematically valid, but we want to choose the one that will lead to a positive k-value. (That way we know that e^{kt} is blowing up and e^{-kt} is approaching zero.) Which of these equations will give us a positive k-value? Will it work for both the hot coffee and the cold soda? How can you tell?

Solution: If $Q > Q_r$, as in the case of the hot cup of coffee, you know that the coffee will cool down, so dQ/dt will be negative. If you choose the differential equation $dQ/dt = k(Q_r - Q)$, then $(Q_r - Q)$ is already negative, so k will be positive. You can also test it for the cold soda, where $Q < Q_r$. The cold soda will warm up, so dQ/dt will be positive. Since $(Q_r - Q)$ is positive, then k will be positive.

Final Answer: $dQ/dt = k(Q_r - Q)$

(b) Solve Newton's Law by separating variables. Note that your function $Q(t)$ will have three constants in it: Q_r (the temperature of the room), k (the constant of proportionality), and C (an arbitrary constant).

Solution:

$$\int \frac{1}{Q_r - Q}\, dQ = \int k\, dt$$
$$-\ln(Q_r - Q) = kt + C$$
$$Q = Q_r - Ce^{-kt}$$

Final Answer: $Q(t) = Q_r - Ce^{-kt}$

(c) What is $\lim_{t\to\infty} Q(t)$?

Final Answer: Q_r

(d) A 90°C cup of coffee placed in a 70°C room reaches 80°C in ten 10 minutes. How long will it take to reach 71°C?

Solution: The initial condition $Q(0) = 90$ gives you the value of the arbitrary constant, $C = -20$. The second condition $Q(10) = 80$ gives you the constant of proportionality, $k = (1/10)(\ln 2)$. Therefore, your solution looks like:

$Q(t) = 70 + 20e^{-(1/10)(\ln 2)t}$

Plugging in $Q = 71$ and solving for t will give you $t = 43$.

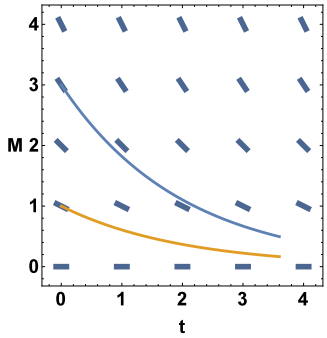
Final Answer: 43 min

1.209 Radioactive Decay

A radioactive sample consists of a large collection of unstable atoms. In any given day (or year or century), every one of those atoms has a certain chance of decaying, independent of all the other atoms. Therefore, the more atoms you have, the more atoms decay: $dM/dt = -kM$ where k is a positive constant.

(a) Draw a slope field for this differential equation on all integer points in $0 \le t \le 4$, $0 \le M \le 4$ using $k = 1/2$. Then draw two sample curves through your slope field.

Solution:



(b) Based on your slope field, what is $\lim_{t\to\infty} M(t)$?

Final Answer: 0

(c) Solve the differential equation with the initial condition $M(0) = M_0$. (Do *not* set $k = 1/2$ as you did in Part a; leave it as an unknown constant that varies from one substance to another.) Does the resulting function match the curves you drew in Part a?

Solution:

$$\int \frac{1}{M}\, dM = \int -k\, dt$$
$$M = Ce^{-kt}$$

Plugging in the initial condition $M(0) = M_0$ gives us $C = M_0$, so the final solution looks like $M(t) = M_0 e^{-kt}$. As time goes to infinity, M approaches 0. So this solution does indeed match the curves we drew in part (a).

Final Answer: $M(t) = M_0 e^{-kt}$

(d) The "half-life" $t_{1/2}$ of a radioactive substance is the amount of time elapsed before $M = M_0/2$. Write a formula for the half-life as a function of k.

Solution: Plugging in $M(t_{1/2}) = M_0/2$ into the solution and solving for $t_{1/2}$, we get $t_{1/2} = (\ln 2)/k$.

Final Answer: $t_{1/2} = (\ln 2)/k$

(e) If you measure that 10% of a radioactive sample has decayed in one day, how long will it be until 90% decays?

Solution: Measuring time in days, this says that $M_0 e^{-k} = 0.9M_0$, so $k = -\ln(0.9)$. When 90% has decayed, we get $M_0 e^{-kt} = 0.1M_0$, so $t = -(1/k)\ln(0.1) = \ln(0.1)/\ln(0.9) \approx 22$, so it will take 22 days.

Final Answer: 22 days

1.210 *[This problem depends on Problem 209.]*

(a) Uranium-235 is used in atomic bombs. How long does it take a sample of ^{235}U to decay to a tenth of its original mass?

Solution: Plugging in $M = M_0/10$ into $M(t) = M_0 e^{-kt}$ and solving for t, you get $t = \ln(10)/k$. To find the k for Uranium-235, you can look up the half-life of Uranium-235 and use part (d) to write $k = (\ln 2)/t_{1/2}$. You will end up with $t = t_{1/2}\ln(10)/(\ln 2)$. The half-life of Uranium-235 is 703.8 million years, so $t = 2,338$ million years.

Final Answer: 2,338 million years

(b) Carbon-14 is used in dating ancient organic samples. How long does it take a sample of ^{14}C to decay to a tenth of its original mass?

Solution: Using the same formula as from part (a) and the fact that $t_{1/2} = 5,700$ years for Carbon-14, you will get $t = 19,000$ years.

Final Answer: 19,000 years

1.211 Rate of a Chemical Reaction

When iron and sulfur are heated together, they can combine to create iron sulfide. Each time an Fe molecule collides with an S molecule, there is a chance that they will form an FeS molecule: **Fe+S→FeS**.

Let f equal the number of g-moles of iron, s equal the number of g-moles of sulfur, and p equal the number of g-moles of iron sulfide produced. The system begins with $f = f_0$, $s = s_0$, and $p = 0$.

(a) Explain why, at any given time, $f = f_0 - p$.

Solution: For every one atom of iron that disappears in the reaction, one molecule of iron sulfide is made. Therefore, the number of g-moles of iron that disappear in the reaction will be the same number of g-moles that appear as iron sulfide.

(b) Suppose at a given moment $df/dt = -5$. What is ds/dt? What is dp/dt?

Solution: df/dt represents the rate at which iron disappears due to the reaction. Since the ratio of the sulfur that's needed to react with iron is 1:1, the rate at which sulfur disappears will be the same as the rate of iron. Hence, $ds/dt = -5$. Also, the ratio of iron sulfide that's made compared to the amount of iron that reacts is also 1:1. So the rate of iron sulfide production will be $dp/dt = 5$.

Final Answer: $ds/dt = -5$ and $dp/dt = 5$

(c) Because the reaction depends upon random collisions between Fe and S molecules, the rate of reaction dp/dt is proportional to the *product* $f \times s$. Write a differential equation for $p(t)$. Note that your differential equation will include the variable p and the constants f_0 and s_0, but cannot include the variable quantities f and s.

Final Answer: $dp/dt = k(f_0 - p)(s_0 - p)$

(d) Solve your differential equation assuming that $f_0 = s_0$.

Solution:

$$\frac{dp}{dt} = k(f_0 - p)^2$$

$$\int \frac{1}{(f_0 - p)^2} dp = \int k\, dt$$

$$\frac{1}{(f_0 - p)} = kt + C$$

$$p(t) = f_0 - \frac{1}{kt + C}$$

Final Answer: $p(t) = f_0 - 1/(kt + C)$

(e) Solve your differential equation assuming that $f_0 \neq s_0$. (*You can solve this on a computer or do it by hand using partial fractions.*)

Solution: Using separation of variables, you get

$$\int \frac{1}{(f_0 - p)(s_0 - p)} dp = \int k\, dt$$

$$\frac{1}{s_0 - f_0} \int \left[\frac{1}{(f_0 - p)} - \frac{1}{(s_0 - p)} \right] dp = kt + C$$

$$\frac{1}{s_0 - f_0} \ln\left(\frac{s_0 - p}{f_0 - p} \right) = kt + C$$

$$\ln\left(\frac{s_0 - p}{f_0 - p} \right) = k(s_0 - f_0)t + C$$

$$\frac{s_0 - p}{f_0 - p} = Ce^{k(s_0 - f_0)t}$$

$$s_0 - p = f_0 Ce^{k(s_0 - f_0)t} - pCe^{k(s_0 - f_0)t}$$

$$p\left(Ce^{k(s_0 - f_0)t} - 1\right) = f_0 Ce^{k(s_0 - f_0)t} - s_0$$

$$p = \frac{f_0 Ce^{k(s_0 - f_0)t} - s_0}{Ce^{k(s_0 - f_0)t} - 1}$$

Final Answer: $p(t) = \dfrac{f_0 Ce^{k(s_0 - f_0)t} - s_0}{Ce^{k(s_0 - f_0)t} - 1}$

(f) Find $\lim_{t \to \infty} p(t)$. The answer will depend on whether $s_0 > f_0$, $s_0 = f_0$, or $s_0 < f_0$.

Solution: For $s_0 = f_0$ we found that:

$$p(t) = f_0 - \frac{1}{kt + C}$$

In this case $\lim_{t \to \infty} p(t) = f_0$.

For $s_0 \neq f_0$ we found that:

$$p(t) = \frac{f_0 Ce^{k(s_0 - f_0)t} - s_0}{Ce^{k(s_0 - f_0)t} - 1}$$

If $s_0 > f_0$ the exponential terms decay. (We are assuming here that $k > 0$. Can you look back at the definition of the constant k and see why that assumption is warranted?) In that case $\lim_{t \to \infty} p(t) = f_0$.

If $s_0 < f_0$ the exponential terms blow up, dominating the constant terms in both the numerator. You can readily see (or you can verify using l'Hospital's rule) in this case that $\lim_{t \to \infty} p(t) = s_0$.

All these results make sense. The smaller of the two reactants is the limiting factor; ultimately it has converted entirely to FeS.

Final Answer: f_0 for $s_0 > f_0$ and $s_0 = f_0$, s_0 for $s_0 < f_0$.

1.212 Spread of a Disease

The *logistic equation* is used to predict, among other things, the spread of disease through a fixed population. For instance, a disease might follow the equation $dP/dt = 2P(1-P)$ where P is the fraction of the population that is infected.

(a) For very low values of P, $1-P$ is approximately 1, so the differential equation is approximately $dP/dt = 2P$. What kind of growth is predicted by this equation? What does it suggest about the mechanism of disease spread?

Solution: This equation predicts exponential growth ($P = Ce^{2t}$), so the disease spreads exponentially fast in its beginning stages. The mechanism is that disease spreads from one contaminated host to others, so the more disease carriers you have, the faster it spreads.

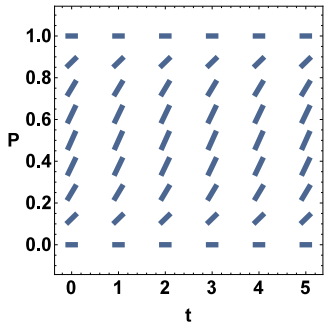
Final Answer: exponential

(b) For very high values of P, $1-P$ approaches zero, so the growth slows down. Why does this make sense in terms of how the disease spreads?

Solution: The disease can only spread to the people who are still not infected, whose numbers are very small. Therefore, the growth is very small.

(c) Draw a slope field for this equation for $P = 0, 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1$. Ignore negative t-values.

Solution:

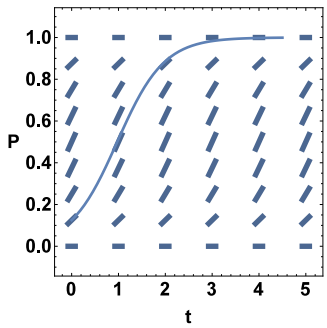


(d) Based on looking at your slope field, list the equilibrium solutions and classify them as stable or unstable.

Final Answer: $P = 0$ unstable, $P = 1$ stable equilibrium

(e) Trace a sample curve through your slope field, starting at $P(0) = 1/8$. Your curve should extend far enough to determine $\lim_{t\to\infty} P(t)$.

Solution:



1.213 *[This problem depends on Problem 212.]*

(a) Solve the differential equation in Problem 212. (*You can solve this on a computer or do it by hand using partial fractions.*)

Solution:

$$\begin{aligned}
\int \frac{1}{P(1-P)}dP &= \int 2\,dt \\
\int \left[\frac{1}{P} + \frac{1}{(1-P)}\right] dP &= \int 2\,dt \\
\ln(P) - \ln(1-P) &= 2t + C \\
\ln\left(\frac{P}{1-P}\right) &= 2t + C \\
\frac{P}{1-P} &= Ce^{2t} \\
P &= Ce^{2t} - PCe^{2t} \\
P\left(Ce^{2t}+1\right) &= Ce^{2t} \\
P(t) &= \frac{Ce^{2t}}{Ce^{2t}+1}
\end{aligned}$$

Final Answer: $P(t) = Ce^{2t}/(Ce^{2t}+1)$

(b) Find the specific solution that corresponds to the initial condition $P(0) = 1/8$.

Final Answer: $P(t) = e^{2t}/(7+e^{2t})$

(c) Find $\lim_{t\to\infty} P(t)$. Make sure it matches your prediction in Problem 212.

Final Answer: 1

(d) 💻 Graph your solution on a computer. Make sure the graph matches the graph you drew based on your slope field.

Solution: