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# CHAPTER 1: SYSTEMS OF LINEAR EQUATIONS AND MATRICES

## 1.1 Introduction to Systems of Linear Equations

**1.** **(a)** This is a linear equation in , , and .

**(b)** This is not a linear equation in , , and because of the term .

**(c)** We can rewrite this equation in the form therefore it is a linear equation in , , and .

**(d)** This is not a linear equation in , , and because of the term .

**(e)** This is not a linear equation in , , and because of the term .

**(f)** This is a linear equation in , , and .

**2.** **(a)** This is a linear equation in and .

**(b)** This is not a linear equation in and because of the terms and .

**(c)** This is a linear equation in and .

**(d)** This is not a linear equation in and because of the term .

**(e)** This is not a linear equation in and because of the term .

**(f)** We can rewrite this equation in the form thus it is a linear equation in and .

**3. (a)**

**(b)**

**(c)**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **4.** | **(a)** |  | **(b)** |  | **(c)** |  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **5.** | **(a)** |  | **(b)** |  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **6.** | **(a)** |  | **(b)** |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **7.** | **(a)** |  | **(b)** |  | **(c)** |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **8.** | **(a)** |  | **(b)** |  | **(c)** |

**9.** The values in (a), (d), and (e) satisfy all three equations – these 3-tuples are solutions of the system.  
The 3-tuples in (b) and (c) are not solutions of the system.

**10.** The values in (b), (d), and (e) satisfy all three equations – these 3-tuples are solutions of the system.  
The 3-tuples in (a) and (c) are not solutions of the system.

**11. (a)** We can eliminate from the second equation by adding times the first equation to the second. This yields the system

The second equation is contradictory, so the original system has no solutions. The lines represented by the equations in that system have no points of intersection (the lines are parallel and distinct).

**(b)** We can eliminate from the second equation by adding times the first equation to the second. This yields the system

The second equation does not impose any restriction on and therefore we can omit it. The lines represented by the original system have infinitely many points of intersection. Solving the first equation for we obtain . This allows us to represent the solution using parametric equations

where the parameter is an arbitrary real number.

**(c)** We can eliminate from the second equation by adding times the first equation to the second. This yields the system

From the second equation we obtain . Substituting for into the first equation results in . Therefore, the original system has the unique solution

The represented by the equations in that system have one point of intersection: .

**12.** We can eliminate from the second equation by adding times the first equation to the second. This yields the system

If (i.e., ) then the second equation imposes no restriction on and ; consequently, the system has infinitely many solutions.

If (i.e., ) then the second equation becomes contradictory thus the system has no solutions.

There are no values of and for which the system has one solution.

**13. (a)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameter is an arbitrary real number.

**(b)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameters and are arbitrary real numbers.

**(c)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameters , , and are arbitrary real numbers.

**(d)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameters , , , and are arbitrary real numbers.

**14. (a)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameter is an arbitrary real number.

**(b)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameters and are arbitrary real numbers.

**(c)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameters , , and are arbitrary real numbers.

**(d)** Solving the equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameters , , , and are arbitrary real numbers.

**15. (a)** We can eliminate from the second equation by adding times the first equation to the second. This yields the system

The second equation does not impose any restriction on and therefore we can omit it. Solving the first equation for we obtain . This allows us to represent the solution using parametric equations

where the parameter is an arbitrary real number.

**(b)** We can see that the second and the third equation are multiples of the first: adding times the first equation to the second, then adding the first equation to the third yields the system

The last two equations do not impose any restriction on the unknowns therefore we can omit them. Solving the first equation for we obtain . This allows us to represent the solution using parametric equations

where the parameters and are arbitrary real numbers.

**16. (a)** We can eliminate from the first equation by adding times the second equation to the first. This yields the system

The first equation does not impose any restriction on and therefore we can omit it. Solving the second equation for we obtain . This allows us to represent the solution using parametric equations

where the parameter is an arbitrary real number.

**(b)** We can see that the second and the third equation are multiples of the first: adding times the first equation to the second, then adding times the first equation to the third yields the system

The last two equations do not impose any restriction on the unknowns therefore we can omit them. Solving the first equation for we obtain . This allows us to represent the solution using parametric equations

where the parameters and are arbitrary real numbers.

**17. (a)** Add times the second row to the first to obtain .

**(b)** Add the third row to the first to obtain

(another solution: interchange the first row and the third row to obtain ).

**18. (a)** Multiply the first row by to obtain .

**(b)** Add the third row to the first to obtain

(another solution: add times the second row to the first to obtain ).

**19. (a)** Add times the first row to the second to obtain which corresponds to the system

If then the second equation becomes , which is contradictory thus the system becomes inconsistent.

If then we can solve the second equation for and proceed to substitute this value into the first equation and solve for .

Consequently, for all values of the given augmented matrix corresponds to a consistent linear system.

**(b)** Add times the first row to the second to obtain which corresponds to the system

If then the second equation becomes , which does not impose any restriction on and therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If then the second equation yields and the first equation becomes .

Consequently, for all values of the given augmented matrix corresponds to a consistent linear system.

**20. (a)** Add times the first row to the second to obtain which corresponds to the system

If then the second equation becomes , which does not impose any restriction on and therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If then the second equation is contradictory thus the system becomes inconsistent.

Consequently, the given augmented matrix corresponds to a consistent linear system only when .

**(b)** Add the first row to the second to obtain which corresponds to the system

If then the second equation becomes , which does not impose any restriction on and therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If then the second equation yields and the first equation becomes .

Consequently, for all values of the given augmented matrix corresponds to a consistent linear system.

**21.** Substituting the coordinates of the first point into the equation of the curve we obtain

Repeating this for the other two points and rearranging the three equations yields

This is a linear system in the unknowns , , and . Its augmented matrix is .

**23.** Solving the first equation for we obtain therefore the solution set of the original equation can be described by the parametric equations

where the parameter is an arbitrary real number.

Substituting these into the second equation yields

which can be rewritten as

This equation must hold true for all real values , which requires that the coefficients associated with the same power of on both sides must be equal. Consequently, and .

**24.** **(a)** The system has no solutions if either

* at least two of the three lines are parallel and distinct or
* each pair of lines intersects at a different point (without any lines being parallel)

**(b)** The system has exactly one solution if either

* two lines coincide and the third one intersects them or
* all three lines intersect at a single point (without any lines being parallel)

**(c)** The system has infinitely many solutions if all three lines coincide.

**25.**

**26.** We set up the linear system as discussed in Exercise 21:

i.e.

One solution is expected, since exactly one parabola passes through any three given points , , if , , and are distinct.

**27.**

True-False Exercises

**(a)** True. is a solution.

**(b)** False. Only multiplication by a **non**zero constant is a valid elementary row operation.

**(c)** True. If then the system has infinitely many solutions; otherwise the system is inconsistent.

**(d)** True. According to the definition, is a linear equation if the 's are not all zero. Let us assume . The values of all 's except for can be set to be arbitrary parameters, and the equation can be used to express in terms of those parameters.

**(e)** False. E.g. if the equations are all homogeneous then the system must be consistent. (See True-False Exercise (a) above.)

**(f)** False. If then the new system has the same solution set as the original one.

**(g)** True. Adding times one row to another amounts to the same thing as subtracting one row from another.

**(h)** False. The second row corresponds to the equation , which is contradictory.