INSTRUCTOR'S SOLUTIONS MANUAL
(ONLINE ONLY)

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PROBABILITY AND STATISTICS
FOURTH EDITION

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Preface

This manual contains solutions to all of the exercises in *Probability and Statistics*, 4th edition, by Morris DeGroot and myself. I have preserved most of the solutions to the exercises that existed in the 3rd edition. Certainly errors have been introduced, and I will post any errors brought to my attention on my web page `http://www.stat.cmu.edu/ mark/` along with errors in the text itself. Feel free to send me comments.

For instructors who are familiar with earlier editions, I hope that you will find the 4th edition at least as useful. Some new material has been added, and little has been removed. Assuming that you will be spending the same amount of time using the text as before, something will have to be skipped. I have tried to arrange the material so that instructors can choose what to cover and what not to cover based on the type of course they want. This manual contains commentary on specific sections right before the solutions for those sections. This commentary is intended to explain special features of those sections and help instructors decide which parts they want to require of their students. Special attention is given to more challenging material and how the remainder of the text does or does not depend upon it.

To teach a mathematical statistics course for students with a strong calculus background, one could safely cover all of the material for which one could find time. The Bayesian sections include 4.8, 7.2, 7.3, 7.4, 8.6, 9.8, and 11.4. One can choose to skip some or all of this material if one desires, but that would be ignoring one of the unique features of the text. The more challenging material in Sections 7.7–7.9, and 9.2–9.4 is really only suitable for a mathematical statistics course. One should try to make time for some of the material in Sections 12.1–12.3 even if it meant cutting back on some of the nonparametrics and two-way ANOVA. To teach a more modern statistics course, one could skip Sections 7.7–7.9, 9.2–9.4, 10.8, and 11.7–11.8. This would leave time to discuss robust estimation (Section 10.7) and simulation (Chapter 12). Section 3.10 on Markov chains is not actually necessary even if one wishes to introduce Markov chain Monte Carlo (Section 12.5), although it is helpful for understanding what this topic is about.

Using Statistical Software

The text was written without reference to any particular statistical or mathematical software. However, there are several places throughout the text where references are made to what general statistical software might be able to do. This is done for at least two reasons. One is that different instructors who wish to use statistical software while teaching will generally choose different programs. I didn't want the text to be tied to a particular program to the exclusion of others. A second reason is that there are still many instructors of mathematical probability and statistics courses who prefer not to use any software at all.

Given how pervasive computing is becoming in the use of statistics, the second reason above is becoming less compelling. Given the free and multiplatform availability and the versatility of the environment R, even the first reason is becoming less compelling. Throughout this manual, I have inserted pointers to which R functions will perform many of the calculations that would formerly have been done by hand when using this text. The software can be downloaded for Unix, Windows, or Mac OS from
`http://www.r-project.org/`
That site also has manuals for installation and use. Help is also available directly from within the R environment.

Many tutorials for getting started with R are available online. At the official R site there is the detailed manual: `http://cran.r-project.org/doc/manuals/R-intro.html`
that starts simple and has a good table of contents and lots of examples. However, reading it from start to finish is *not* an efficient way to get started. The sample sessions should be most helpful.

One major issue with using an environment like R is that it is essentially programming. That is, students who have never programmed seriously before are going to have a steep learning curve. Without going into the philosophy of whether students should learn statistics without programming, the field is moving in the direction of requiring programming skills. People who want only to understand what a statistical analysis

is about can still learn that without being able to program. But anyone who actually wants to do statistics as part of their job will be seriously handicapped without programming ability. At the end of this manual is a series of heavily commented R programms that illustrate many of the features of R in the context of a specific example from the text.

Mark J. Schervish

Chapter 1

Introduction to Probability

1.2 Interpretations of Probability

Commentary

It is interesting to have the students determine some of their own subjective probabilities. For example, let X denote the temperature at noon tomorrow outside the building in which the class is being held. Have each student determine a number x_1 such that the student considers the following two possible outcomes to be equally likely: $X \le x_1$ and $X > x_1$. Also, have each student determine numbers x_2 and x_3 (with $x_2 < x_3$) such that the student considers the following three possible outcomes to be equally likely: $X \le x_2$, $x_2 < X < x_3$, and $X \ge x_3$. Determinations of more than three outcomes that are considered to be equally likely can also be made. The different values of x_1 determined by different members of the class should be discussed, and also the possibility of getting the class to agree on a common value of x_1.

Similar determinations of equally likely outcomes can be made by the students in the class for quantities such as the following ones which were found in the 1973 World Almanac and Book of Facts: the number of freight cars that were in use by American railways in 1960 (1,690,396), the number of banks in the United States which closed temporarily or permanently in 1931 on account of financial difficulties (2,294), and the total number of telephones which were in service in South America in 1971 (6,137,000).

1.4 Set Theory

Solutions to Exercises

1. Assume that $x \in B^c$. We need to show that $x \in A^c$. We shall show this indirectly. Assume, to the contrary, that $x \in A$. Then $x \in B$ because $A \subset B$. This contradicts $x \in B^c$. Hence $x \in A$ is false and $x \in A^c$.

2. First, show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. That is, $x \in A$ and either $x \in B$ or $x \in C$ (or both). So either ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) or both. That is, either $x \in A \cap B$ or $x \in A \cap C$. This is what it means to say that $x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Basically, running these steps backwards shows that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

3. To prove the first result, let $x \in (A \cup B)^c$. This means that x is not in $A \cup B$. In other words, x is neither in A nor in B. Hence $x \in A^c$ and $x \in B^c$. So $x \in A^c \cap B^c$. This proves that $(A \cup B)^c \subset A^c \cap B^c$. Next, suppose that $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. So x is neither in A nor in B, so it can't be in $A \cup B$. Hence $x \in (A \cup B)^c$. This shows that $A^c \cap B^c \subset (A \cup B)^c$. The second result follows from the first by applying the first result to A^c and B^c and then taking complements of both sides.

4. To see that $A \cap B$ and $A \cap B^c$ are disjoint, let $x \in A \cap B$. Then $x \in B$, hence $x \notin B^c$ and so $x \notin A \cap B^c$. So no element of $A \cap B$ is in $A \cap B^c$, hence the two events are disjoint. To prove that $A = (A \cap B) \cup (A \cap B^c)$, we shall show that each side is a subset of the other side. First, let $x \in A$. Either $x \in B$ or $x \in B^c$. If $x \in B$, then $x \in A \cap B$. If $x \in B^c$, then $x \in A \cap B^c$. Either way, $x \in (A \cap B) \cup (A \cap B^c)$. So every element of A is an element of $(A \cap B) \cup (A \cap B^c)$ and we conclude that $A \subset (A \cap B) \cup (A \cap B^c)$. Finally, let $x \in (A \cap B) \cup (A \cap B^c)$. Then either $x \in A \cap B$, in which case $x \in A$, or $x \in A \cap B^c$, in which case $x \in A$. Either way $x \in A$, so every element of $(A \cap B) \cup (A \cap B^c)$ is also an element of A and $(A \cap B) \cup (A \cap B^c) \subset A$.

5. To prove the first result, let $x \in (\cup_i A_i)^c$. This means that x is not in $\cup_i A_i$. In other words, for every $i \in I$, x is not in A_i. Hence for every $i \in I$, $x \in A_i^c$. So $x \in \cap_i A_i^c$. This proves that $(\cup_i A_i)^c \subset \cap_i A_i^c$. Next, suppose that $x \in \cap_i A_i^c$. Then $x \in A_i^c$ for every $i \in I$. So for every $i \in I$, x is not in A_i. So x can't be in $\cup_i A_i$. Hence $x \in (\cup_i A_i)^c$. This shows that $\cap_i A_i^c \subset (\cup_i A_i)^c$. The second result follows from the first by applying the first result to A_i^c for $i \in I$ and then taking complements of both sides.

6. (a) Blue card numbered 2 or 4.
   (b) Blue card numbered 5, 6, 7, 8, 9, or 10.
   (c) Any blue card or a red card numbered 1, 2, 3, 4, 6, 8, or 10.
   (d) Blue card numbered 2, 4, 6, 8, or 10, or red card numbered 2 or 4.
   (e) Red card numbered 5, 7, or 9.

7. (a) These are the points not in A, hence they must be either below 1 or above 5. That is $A^c = \{x : x < 1 \text{ or } x > 5\}$.
   (b) These are the points in either A or B or both. So they must be between 1 and 5 or between 3 and 7. That is, $A \cup B = \{x : 1 \le x \le 7\}$.
   (c) These are the points in B but not in C. That is $BC^c = \{x : 3 < x \le 7\}$. (Note that $B \subset C^c$.)
   (d) These are the points in none of the three sets, namely $A^cB^cC^c = \{x : 0 < x < 1 \text{ or } x > 7\}$.
   (e) These are the points in the answer to part (b) and in C. There are no such values and $(A \cup B)C = \emptyset$.

8. Blood type A reacts only with anti-A, so type A blood corresponds to $A \cap B^c$. Type B blood reacts only with anti-B, so type B blood corresponds to A^cB. Type AB blood reacts with both, so $A \cap B$ characterizes type AB blood. Finally, type O reacts with neither antigen, so type O blood corresponds to the event A^cB^c.

9. (a) For each n, $B_n = B_{n+1} \cup A_n$, hence $B_n \supset B_{n+1}$ for all n. For each n, $C_{n+1} \cap A_n = C_n$, so $C_n \subset C_{n+1}$.
   (b) Suppose that $x \in \cap_{n=1}^{\infty} B_n$. Then $x \in B_n$ for all n. That is, $x \in \cup_{i=n}^{\infty} A_i$ for all n. For $n = 1$, there exists $i \ge n$ such that $x \in A_i$. Assume to the contrary that there are at most finitely many i such that $x \in A_i$. Let m be the largest such i. For $n = m + 1$, we know that there is $i \ge n$ such that $x \in A_i$. This contradicts m being the largest i such that $x \in A_i$. Hence, x is in infinitely many A_i. For the other direction, assume that x is in infinitely many A_i. Then, for every n, there is a value of $j > n$ such that $x \in A_j$, hence $x \in \cup_{i=n}^{\infty} A_i = B_n$ for every n and $x \in \cap_{n=1}^{\infty} B_n$.
   (c) Suppose that $x \in \cup_{n=1}^{\infty} C_n$. That is, there exists n such that $x \in C_n = \cap_{i=n}^{\infty} A_i$, so $x \in A_i$ for all $i \ge n$. So, there at most finitely many i (a subset of $1, \ldots, n-1$) such that $x \notin A_i$. Finally, suppose that $x \in A_i$ for all but finitely many i. Let k be the last i such that $x \notin A_i$. Then $x \in A_i$ for all $i \ge k + 1$, hence $x \in \cap_{i=k+1}^{\infty} A_i = C_{k+1}$. Hence $x \in \cup_{n=1}^{\infty} C_n$.

10. (a) All three dice show even numbers if and only if all three of A, B, and C occur. So, the event is $A \cap B \cap C$.
(b) None of the three dice show even numbers if and only if all three of A^c, B^c, and C^c occur. So, the event is $A^c \cap B^c \cap C^c$.
(c) At least one die shows an odd number if and only if at least one of A^c, B^c, and C^c occur. So, the event is $A^c \cup B^c \cup C^c$.
(d) At most two dice show odd numbers if and only if at least one die shows an even number, so the event is $A \cup B \cup C$. This can also be expressed as the union of the three events of the form $A \cap B \cap C^c$ where exactly one die shows odd together with the three events of the form $A \cap B^c \cap C^c$ where exactly two dice show odd together with the even $A \cap B \cap C$ where no dice show odd.
(e) We can enumerate all the sums that are no greater than 5: $1+1+1$, $2+1+1$, $1+2+1$, $1+1+2$, $2+2+1$, $2+1+2$, and $1+2+2$. The first of these corresponds to the event $A_1 \cap B_1 \cap C_1$, the second to $A_2 \cap B_1 \cap C_1$, etc. The union of the seven such events is what is requested, namely

$$(A_1 \cap B_1 \cap C_1) \cup (A_2 \cap B_1 \cap C_1) \cup (A_1 \cap B_2 \cap C_1) \cup (A_1 \cap B_1 \cap C_2) \cup (A_2 \cap B_2 \cap C_1) \cup (A_2 \cap B_1 \cap C_2) \cup (A_1 \cap B_2 \cap C_2).$$

11. (a) All of the events mentioned can be determined by knowing the voltages of the two subcells. Hence the following set can serve as a sample space

$$S = \{(x, y) : 0 \le x \le 5 \text{ and } 0 \le y \le 5\},$$

where the first coordinate is the voltage of the first subcell and the second coordinate is the voltage of the second subcell. Any more complicated set from which these two voltages can be determined could serve as the sample space, so long as each outcome could at least hypothetically be learned.
(b) The power cell is functional if and only if the sum of the voltages is at least 6. Hence, $A = \{(x, y) \in S : x + y \ge 6\}$. It is clear that $B = \{(x, y) \in S : x = y\}$ and $C = \{(x, y) \in S : x > y\}$. The powercell is not functional if and only if the sum of the voltages is less than 6. It needs less than one volt to be functional if and only if the sum of the voltages is greater than 5. The intersection of these two is the event $D = \{(x, y) \in S : 5 < x + y < 6\}$. The restriction "$\in S$" that appears in each of these descriptions guarantees that the set is a subset of S. One could leave off this restriction and add the two restrictions $0 \le x \le 5$ and $0 \le y \le 5$ to each set.
(c) The description can be worded as "the power cell is not functional, and needs at least one more volt to be functional, and both subcells have the same voltage." This is the intersection of A^c, D^c, and B. That is, $A^c \cap D^c \cap B$. The part of D^c in which $x + y \ge 6$ is not part of this set because of the intersection with A^c.
(d) We need the intersection of A^c (not functional) with C^c (second subcell at least as big as first) and with B^c (subcells are not the same). In particular, $C^c \cap B^c$ is the event that the second subcell is strictly higher than the first. So, the event is $A^c \cap B^c \cap C^c$.

1.5 The Definition of Probability

Solutions to Exercises

1. Define the following events:

$$\begin{aligned} A &= \{\text{the selected ball is red}\}, \\ B &= \{\text{the selected ball is white}\}, \\ C &= \{\text{the selected ball is either blue, yellow, or green}\}. \end{aligned}$$

We are asked to find $\Pr(C)$. The three events A, B, and C are disjoint and $A \cup B \cup C = S$. So $1 = \Pr(A) + \Pr(B) + \Pr(C)$. We are told that $\Pr(A) = 1/5$ and $\Pr(B) = 2/5$. It follows that $\Pr(C) = 2/5$.

2. Let B be the event that a boy is selected, and let G be the event that a girl is selected. We are told that $B \cup G = S$, so $G = B^c$. Since $\Pr(B) = 0.3$, it follows that $\Pr(G) = 0.7$.

3. (a) If A and B are disjoint then $B \subset A^c$ and $BA^c = B$, so $\Pr(BA^c) = \Pr(B) = 1/2$.
   (b) If $A \subset B$, then $B = A \cup (BA^c)$ with A and BA^c disjoint. So $\Pr(B) = \Pr(A) + \Pr(BA^c)$. That is, $1/2 = 1/3 + \Pr(BA^c)$, so $\Pr(BA^c) = 1/6$.
   (c) According to Theorem 1.4.11, $B = (BA) \cup (BA^c)$. Also, BA and BA^c are disjoint so, $\Pr(B) = \Pr(BA) + \Pr(BA^c)$. That is, $1/2 = 1/8 + \Pr(BA^c)$, so $\Pr(BA^c) = 3/8$.

4. Let E_1 be the event that student A fails and let E_2 be the event that student B fails. We want $\Pr(E_1 \cup E_2)$. We are told that $\Pr(E_1) = 0.5$, $\Pr(E_2) = 0.2$, and $\Pr(E_1E_2) = 0.1$. According to Theorem 1.5.7, $\Pr(E_1 \cup E_2) = 0.5 + 0.2 - 0.1 = 0.6$.

5. Using the same notation as in Exercise 4, we now want $\Pr(E_1^c \cap E_2^c)$. According to Theorems 1.4.9 and 1.5.3, this equals $1 - \Pr(E_1 \cup E_2) = 0.4$.

6. Using the same notation as in Exercise 4, we now want $\Pr([E_1 \cap E_2^c] \cup [E_1^c \cap E_2])$. These two events are disjoint, so

$$\Pr([E_1 \cap E_2^c] \cup [E_1^c \cap E_2]) = \Pr(E_1 \cap E_2^c) + \Pr(E_1^c \cap E_2).$$

   Use the reasoning from part (c) of Exercise 3 above to conclude that

$$\begin{aligned} \Pr(E_1 \cap E_2^c) &= \Pr(E_1) - \Pr(E_1 \cap E_2) = 0.4, \\ \Pr(E_1^c \cap E_2) &= \Pr(E_2) - \Pr(E_1 \cap E_2) = 0.1. \end{aligned}$$

   It follows that the probability we want is 0.5.

7. Rearranging terms in Eq. (1.5.1) of the text, we get

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.4 + 0.7 - \Pr(A \cup B) = 1.1 - \Pr(A \cup B).$$

   So $\Pr(A \cap B)$ is largest when $\Pr(A \cup B)$ is smallest and vice-versa. The smallest possible value for $\Pr(A \cup B)$ occurs when one of the events is a subset of the other. In the present exercise this could only happen if $A \subset B$, in which case $\Pr(A \cup B) = \Pr(B) = 0.7$, and $\Pr(A \cap B) = 0.4$. The largest possible value of $\Pr(A \cup B)$ occurs when either A and B are disjoint or when $A \cup B = S$. The former is not possible since the probabilities are too large, but the latter is possible. In this case $\Pr(A \cup B) = 1$ and $\Pr(A \cap B) = 0.1$.

8. Let A be the event that a randomly selected family subscribes to the morning paper, and let B be the event that a randomly selected family subscribes to the afternoon paper. We are told that $\Pr(A) = 0.5$, $\Pr(B) = 0.65$, and $\Pr(A \cup B) = 0.85$. We are asked to find $\Pr(A \cap B)$. Using Theorem 1.5.7 in the text we obtain

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.5 + 0.65 - 0.85 = 0.3.$$

9. The required probability is

$$\begin{aligned}\Pr(A \cap B^C) + \Pr(A^C B) &= [\Pr(A) - \Pr(A \cap B)] + [\Pr(B) - \Pr(A \cap B)] \\ &= \Pr(A) + \Pr(B) - 2\Pr(A \cap B).\end{aligned}$$

10. Theorem 1.4.11 says that $A = (A \cap B) \cup (A \cap B^c)$. Clearly the two events $A \cap B$ and $A \cap B^c$ are disjoint. It follows from Theorem 1.5.6 that $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c)$.

11. (a) The set of points for which $(x - 1/2)^2 + (y - 1/2)^2 < 1/4$ is the interior of a circle that is contained in the unit square. (Its center is $(1/2, 1/2)$ and its radius is $1/2$.) The area of this circle is $\pi/4$, so the area of the remaining region (what we want) is $1 - \pi/4$.

    (b) We need the area of the region between the two lines $y = 1/2 - x$ and $y = 3/2 - x$. The remaining area is the union of two right triangles with base and height both equal to $1/2$. Each triangle has area $1/8$, so the region between the two lines has area $1 - 2/8 = 3/4$.

    (c) We can use calculus to do this. We want the area under the curve $y = 1 - x^2$ between $x = 0$ and $x = 1$. This equals

    $$\int_0^1 (1 - x^2)dx = x - \left.\frac{x^3}{3}\right|_{x=0}^{1} = \frac{2}{3}.$$

    (d) The area of a line is 0, so the probability of a line segment is 0.

12. The events $B_1, B_2, \ldots$ are disjoint, because the event B_1 contains the points in A_1, the event B_2 contains the points in A_2 but not in A_1, the event B_3 contains the points in A_3 but not in A_1 or A_2, etc. By this same reasoning, it is seen that $\cup_{i=1}^n A_i = \cup_{i=1}^n B_i$ and $\cup_{i=1}^\infty A_i = \cup_{i=1}^\infty B_i$. Therefore,

    $$\Pr\left(\bigcup_{i=1}^n A_i\right) = \Pr\left(\bigcup_{i=1}^n B_i\right)$$

    and

    $$\Pr\left(\bigcup_{i=1}^\infty A_i\right) = \Pr\left(\bigcup_{i=1}^\infty B_i\right).$$

    However, since the events $B_1, B_2, \ldots$ are disjoint,

    $$\Pr\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n Pr(B_i)$$

    and

    $$\Pr\left(\bigcup_{i=1}^\infty B_i\right) = \sum_{i=1}^\infty \Pr(B_i).$$

13. We know from Exercise 12 that

    $$\Pr\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \Pr(B_i).$$

Furthermore, from the definition of the events $B_1, \ldots, B_n$ it is seen that $B_i \subset A_i$ for $i = 1, \ldots, n$. Therefore, by Theorem 1.5.4, $\Pr(B_i) \le \Pr(A_i)$ for $i = 1, \ldots, n$. It now follows that

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \Pr(A_i).$$

(Of course, if the events $A_1, \ldots, A_n$ are disjoint, there is equality in this relation.)

For the second part, apply the first part with A_i replaced by A_i^c for $i = 1, \ldots, n$. We get

$$\Pr\left(\bigcup A_i^c\right) \le \sum_{i=1}^{n} \Pr(A_i^c). \tag{S.1.1}$$

Exercise 5 in Sec. 1.4 says that the left side of (S.1.1) is $\Pr\left([\bigcap A_i]^c\right)$. Theorem 1.5.3 says that this last probability is $1 - \Pr\left(\bigcap A_i\right)$. Hence, we can rewrite (S.1.1) as

$$1 - \Pr\left(\bigcap A_i\right) \le \sum_{i=1}^{n} \Pr(A_i^c).$$

Finally take one minus both sides of the above inequality (which reverses the inequality) and produces the desired result.

14. First, note that the probability of type AB blood is $1-(0.5+0.34+0.12) = 0.04$ by using Theorems 1.5.2 and 1.5.3.

    (a) The probability of blood reacting to anti-A is the probability that the blood is either type A or type AB. Since these are disjoint events, the probability is the sum of the two probabilities, namely $0.34 + 0.04 = 0.38$. Similarly, the probability of reacting with anti-B is the probability of being either type B or type AB, $0.12 + 0.04 = 0.16$.

    (b) The probability that both antigens react is the probability of type AB blood, namely 0.04.

1.6 Finite Sample Spaces

Solutions to Exercises

1. The safe way to obtain the answer at this stage of our development is to count that 18 of the 36 outcomes in the sample space yield an odd sum. Another way to solve the problem is to note that regardless of what number appears on the first die, there are three numbers on the second die that will yield an odd sum and three numbers that will yield an even sum. Either way the probability is 1/2.

2. The event whose probability we want is the complement of the event in Exercise 1, so the probability is also 1/2.

3. The only differences greater than or equal to 3 that are available are 3, 4 and 5. These large difference only occur for the six outcomes in the upper right and the six outcomes in the lower left of the array in Example 1.6.5 of the text. So the probability we want is $1 - 12/36 = 2/3$.

4. Let x be the proportion of the school in grade 3 (the same as grades 2–6). Then $2x$ is the proportion in grade 1 and $1 = 2x + 5x = 7x$. So $x = 1/7$, which is the probability that a randomly selected student will be in grade 3.

5. The probability of being in an odd-numbered grade is $2x + x + x = 4x = 4/7$.

6. Assume that all eight possible combinations of faces are equally likely. Only two of those combinations have all three faces the same, so the probability is $1/4$.

7. The possible genotypes of the offspring are aa and Aa, since one parent will definitely contribute an a, while the other can contribute either A or a. Since the parent who is Aa contributes each possible allele with probability $1/2$ each, the probabilities of the two possible offspring are each $1/2$ as well.

8. (a) The sample space contains 12 outcomes: (Head, 1), (Tail, 1), (Head, 2), (Tail, 2), etc.
   (b) Assume that all 12 outcomes are equally likely. Three of the outcomes have Head and an odd number, so the probability is $1/4$.

1.7 Counting Methods

Commentary

If you wish to stress computer evaluation of probabilities, then there are programs for computing factorials and log-factorials. For example, in the statistical software R, there are functions `factorial` and `lfactorial` that compute these. If you cover Stirling's formula (Theorem 1.7.5), you can use these functions to illustrate the closeness of the approximation.

Solutions to Exercises

1. Each pair of starting day and leap year/no leap year designation determines a calendar, and each calendar correspond to exactly one such pair. Since there are seven days and two designations, there are a total of $7 \times 2 = 14$ different calendars.

2. There are 20 ways to choose the student from the first class, and no matter which is chosen, there are 18 ways to choose the student from the second class. No matter which two students are chosen from the first two classes, there are 25 ways to choose the student from the third class. The multiplication rule can be applied to conclude that the total number of ways to choose the three members is $20 \times 18 \times 25 = 9000$.

3. This is a simple matter of permutations of five distinct items, so there are $5! = 120$ ways.

4. There are six different possible shirts, and no matter what shirt is picked, there are four different slacks. So there are 24 different combinations.

5. Let the sample space consist of all four-tuples of dice rolls. There are $6^4 = 1296$ possible outcomes. The outcomes with all four rolls different consist of all of the permutations of six items taken four at a time. There are $P_{6,4} = 360$ of these outcomes. So the probability we want is $360/1296 = 5/18$.

6. With six rolls, there are $6^6 = 46656$ possible outcomes. The outcomes with all different rolls are the permutations of six distinct items. There are $6! = 720$ outcomes in the event of interest, so the probability is $720/46656 = 0.01543$.

7. There are 20^{12} possible outcomes in the sample space. If the 12 balls are to be thrown into different boxes, the first ball can be thrown into any one of the 20 boxes, the second ball can then be thrown into any one of the other 19 boxes, etc. Thus, there are $20 \cdot 19 \cdot 18 \cdots 9$ possible outcomes in the event. So the probability is $20!/[8!20^{12}]$.

8. There are 7^5 possible outcomes in the sample space. If the five passengers are to get off at different floors, the first passenger can get off at any one of the seven floors, the second passenger can then get off at any one of the other six floors, etc. Thus, the probability is

$$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5} = \frac{360}{2401}.$$

9. There are 6! possible arrangements in which the six runners can finish the race. If the three runners from team A finish in the first three positions, there are 3! arrangements of these three runners among these three positions and there are also 3! arrangements of the three runners from team B among the last three positions. Therefore, there are 3! $\times$ 3! arrangements in which the runners from team A finish in the first three positions and the runners from team B finish in the last three positions. Thus, the probability is $(3!3!)/6! = 1/20$.

10. We can imagine that the 100 balls are randomly ordered in a list, and then drawn in that order. Thus, the required probability in part (a), (b), or (c) of this exercise is simply the probability that the first, fiftieth, or last ball in the list is red. Each of these probabilities is the same $\frac{r}{100}$, because of the random order of the list.

11. In terms of factorials, $P_{n,k} = n!/[k!(n-k)!]$. Since we are assuming that n and $n = k$ are large, we can use Stirling's formula to approximate both of them. The approximation to $n!$ is $(2\pi)^{1/2} n^{n+1/2} e^{-n}$, and the approximation to $(n-k)!$ is $(2\pi)^{1/2}(n-k)^{n-k+1/2}e^{-n+k}$. The approximation to the ratio is the ratio of the approximations because the ratio of each approximation to its corresponding factorial converges to 1. That is,

$$\frac{n!}{k!(n-k)!} \approx \frac{(2\pi)^{1/2} n^{n+1/2} e^{-n}}{k!(2\pi)^{1/2}(n-k)^{n-k+1/2}e^{-n+k}} = \frac{e^{-k}n^k}{k!}\left(1 - \frac{k}{n}\right)^{-n-k-1/2}.$$

Further simplification is available if one assumes that k is small compared to n, that is $k/n \approx 0$. In this case, the last factor is approximately e^k, and the whole approximation simplifies to $n^k/k!$. This makes sense because, if $n/(n-k)$ is essentially 1, then the product of the k largest factors in $n!$ is essentially n^k.

1.8 Combinatorial Methods

Commentary

This section ends with an extended example called "The Tennis Tournament". This is an application of combinatorics that uses a slightly subtle line of reasoning.

Solutions to Exercises

1. We have to assign 10 houses to one pollster, and the other pollster will get to canvas the other 10 houses. Hence, the number of assignments is the number of combinations of 20 items taken 10 at a time,

$$\binom{20}{10} = 184{,}756.$$

2. The ratio of $\binom{93}{30}$ to $\binom{93}{31}$ is $31/63 < 1$, so $\binom{93}{31}$ is larger.

3. Since $93 = 63 + 30$, the two numbers are the same.

4. Let the sample space consist of all subsets (not ordered tuples) of the 24 bulbs in the box. There are $\binom{24}{4} = 10626$ such subsets. There is only one subset that has all four defectives, so the probability we want is $1/10626$.

5. The number is $\dfrac{4251!}{(97!4154!)} = \binom{4251}{97}$, an integer.

6. There are $\binom{n}{2}$ possible pairs of seats that A and B can occupy. Of these pairs, $n-1$ pairs comprise two adjacent seats. Therefore, the probability is $\dfrac{n-1}{\binom{n}{2}} = \dfrac{2}{n}$.

7. There are $\binom{n}{k}$ possible sets of k seats to be occupied, and they are all equally likely. There are $n-k+1$ sets of k adjacent seats, so the probability we want is

$$\frac{n-k+1}{\binom{n}{k}} = \frac{(n-k+1)!k!}{n!}.$$

8. There are $\binom{n}{k}$ possible sets of k seats to be occupied, and they are all equally likely. Because the circle has no start or end, there are n sets of k adjacent seats, so the probability we want is

$$\frac{n}{\binom{n}{k}} = \frac{(n-k)!k!}{(n-1)!}.$$

9. This problem is slightly tricky. The total number of ways of choosing the n seats that will be occupied by the n people is $\binom{2n}{n}$. Offhand, it would seem that there are only two ways of choosing these seats so that no two adjacent seats are occupied, namely:

$$\text{X0X0}\ldots0 \quad \text{and} \quad \text{0X0X}\ldots\text{0X}$$

Upon further consideration, however, $n-1$ more ways can be found, namely:

$$\text{X00X0X}\ldots\text{0X}, \quad \text{X0X00X0X}\ldots\text{0X, etc.}$$

Therefore, the total number of ways of choosing the seats so that no two adjacent seats are occupied is $n+1$. The probability is $(n+1)/\binom{2n}{n}$.

10. We shall let the sample space consist of all subsets (unordered) of 10 out of the 24 light bulbs in the box. There are $\binom{24}{10}$ such subsets. The number of subsets that contain the two defective bulbs is the number of subsets of size 8 out of the other 22 bulbs, $\binom{22}{8}$, so the probability we want is

$$\frac{\binom{22}{8}}{\binom{24}{10}} = \frac{10 \times 9}{24 \times 23} = 0.1630.$$

11. This exercise is similar to Exercise 10. Let the sample space consist of all subsets (unordered) of 12 out of the 100 people in the group. There are $\binom{100}{12}$ such subsets. The number of subsets that contain A and B is the number of subsets of size 10 out of the other 98 people, $\binom{98}{10}$, so the probability we want is

$$\frac{\binom{98}{10}}{\binom{100}{12}} = \frac{12 \times 11}{100 \times 99} = 0.01333.$$

12. There are $\binom{35}{10}$ ways of dividing the group into the two teams. As in Exercise 11, the number of ways of choosing the 10 players for the first team so as to include both A and B is $\binom{33}{8}$. The number of ways of choosing the 10 players for this team so as not to include either A or B (A and B will then be together on the other team) is $\binom{33}{10}$. The probability we want is then

$$\frac{\binom{33}{8} + \binom{33}{10}}{\binom{35}{10}} = \frac{10 \times 9 + 25 \times 24}{35 \times 34} = 0.5798.$$

13. This exercise is similar to Exercise 12. Here, we want four designated bulbs to be in the same group. The probability is

$$\frac{\binom{20}{6} + \binom{20}{10}}{\binom{24}{10}} = 0.1140.$$

14.

$$\begin{aligned}\binom{n}{k}+\binom{n}{k-1} &= \frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!}\\ &= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k}+\frac{1}{n-k+1}\right)\\ &= \frac{n!}{(k-1)!(n-k)!}\cdot\frac{n+1}{k(n-k+1)}\\ &= \frac{(n+1)!}{k!(n-k+1)!}=\binom{n+1}{k}.\end{aligned}$$

15. (a) If we express 2^n as $(1+1)^n$ and expand $(1+1)^n$ by the binomial theorem, we obtain the desired result.

(b) If we express 0 as $(1-1)^n$ and expand $(1-1)^n$ by the binomial theorem, we obtain the desired result.

16. (a) It is easier to calculate first the probability that the committee will not contain either of the two senators from the designated state. This probability is $\binom{98}{8}\Big/\binom{100}{8}$. Thus, the final answer is

$$1-\frac{\binom{98}{8}}{\binom{100}{8}}\approx 1-.08546=0.1543.$$

(b) There are $\binom{100}{50}$ combinations that might be chosen. If the group is to contain one senator from each state, then there are two possible choices for each of the fifty states. Hence, the number of possible combinations containing one senator from each state is 2^{50}.

17. Call the four players A, B, C, and D. The number of ways of choosing the positions in the deck that will be occupied by the four aces is $\binom{52}{4}$. Since player A will receive 13 cards, the number of ways of choosing the positions in the deck for the four aces so that all of them will be received by player A is $\binom{13}{4}$. Similarly, since player B will receive 13 other cards, the number of ways of choosing the positions for the four aces so that all of them will be received by player B is $\binom{13}{4}$. A similar result is true for each of the other players. Therefore, the total number of ways of choosing the positions in the deck for the four aces so that all of them will be received by the same player is $4\binom{13}{4}$. Thus, the final probability is $4\binom{13}{4}\Big/\binom{52}{4}$.

18. There are $\binom{100}{10}$ ways of choosing ten mathematics students. There are $\binom{20}{2}$ ways of choosing two

students from a given class of 20 students. Therefore, there are $\binom{20}{2}^5$ ways of choosing two students from each of the five classes. So, the final answer is $\binom{20}{2}^5 / \binom{100}{10} \approx 0.0143$.

19. From the description of what counts as a collection of customer choices, we see that each collection consists of a tuple $(m_1, \ldots, m_n)$, where m_i is the number of customers who choose item i for $i = 1, \ldots, n$. Each m_i must be between 0 and k and $m_1 + \cdots + m_n = k$. Each such tuple is equivalent to a sequence of $n + k - 1$ 0's and 1's as follows. The first m_1 terms are 0 followed by a 1. The next m_2 terms are 0 followed by a 1, and so on up to m_{n-1} 0's followed by a 1 and finally m_n 0's. Since $m_1 + \cdots + m_n = k$ and since we are putting exactly $n - 1$ 1's into the sequence, each such sequence has exactly $n + k - 1$ terms. Also, it is clear that each such sequence corresponds to exactly one tuple of customer choices. The numbers of 0's between successive 1's give the numbers of customers who choose that item, and the 1's indicate where we switch from one item to the next. So, the number of combinations of choices is the number of such sequences: $\binom{n+k-1}{k}$.

20. We shall use induction. For $n = 1$, we must prove that

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0.$$

Since the right side of this equation is $x + y$, the theorem is true for $n = 1$. Now assume that the theorem is true for each $n = 1, \ldots, n_0$ for $n_0 \geq 1$. For $n = n_0 + 1$, the theorem says

$$(x+y)^{n_0+1} = \sum_{k=0}^{n_0+1} \binom{n_0+1}{k} x^k y^{n_0+1-k}. \tag{S.1.2}$$

Since we have assumed that the theorem is true for $n = n_0$, we know that

$$(x+y)^{n_0} = \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0-k}. \tag{S.1.3}$$

We shall multiply both sides of (S.1.3) by $x + y$. We then need to prove that $x + y$ times the right side of (S.1.3) equals the right side of (S.1.2).

$$\begin{aligned}
(x+y)(x+y)^{n_0} &= (x+y) \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0-k} \\
&= \sum_{k=0}^{n_0} \binom{n_0}{k} x^{k+1} y^{n_0-k} + \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0+1-k} \\
&= \sum_{k=1}^{n_0+1} \binom{n_0}{k-1} x^k y^{n_0+1-k} + \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0+1-k} \\
&= y^{n_0+1} + \sum_{k=1}^{n_0} \left[\binom{n_0}{k-1} + \binom{n_0}{k} \right] x^k y^{n_0+1-k} + x^{n_0+1}.
\end{aligned}$$

Now, apply the result in Exercise 14 to conclude that

$$\binom{n_0}{k-1} + \binom{n_0}{k} = \binom{n_0+1}{k}.$$

This makes the final summation above equal to the right side of (S.1.2).

21. We are asked for the number of unordered samples with replacement, as constructed in Exercise 19. Here, $n = 365$, so there are $\binom{365+k}{k}$ different unordered sets of k birthdays chosen with replacement from $1, \ldots, 365$.

22. The approximation to $n!$ is $(2\pi)^{1/2} n^{n+1/2} e^{-n}$, and the approximation to $(n/2)!$ is $(2\pi)^{1/2}(n/2)^{n/2+1/2}e^{-n/2}$. Then

$$\frac{n!}{(n/2)!^2} \approx \frac{(2\pi)^{1/2} n^{n+1/2} e^{-n}}{[(2\pi)^{1/2}(n/2)^{n/2+1/2}e^{-n/2}]^2} = (2\pi)^{-1/2} 2^{n+1} n^{-1/2}.$$

With $n = 500$, the approximation is $e^{343.24}$, too large to represent on a calculator with only two-digit exponents. The actual number is about 1/20 of 1% larger.

1.9 Multinomial Coefficients

Commentary

Multinomial coefficients are useful as a counting method, and they are needed for the definition of the multinomial distribution in Sec. 5.9. They are not used much elsewhere in the text. Although this section does not have an asterisk, it could be skipped (together with Sec. 5.9) if one were not interested in the multinomial distribution or the types of counting arguments that rely on multinomial coefficients.

Solutions to Exercises

1. We have three types of elements that need to be assigned to 21 houses so that exactly seven of each type are assigned. The number of ways to do this is the multinomial coefficient

$$\binom{21}{7,7,7} = 399{,}072{,}960.$$

2. We are asked for the number of arrangements of four distinct types of objects with 18 or one type, 12 of the next, 8 of the next and 12 of the last. This is the multinomial coefficient $\binom{50}{18, 12, 8, 12}$.

3. We need to divide the 300 members of the organization into three subsets: the 5 in one committee, the 8 in the second committee, and the 287 in neither committee. There are $\binom{300}{5, 8, 287}$ ways to do this.

4. There are $\binom{10}{3, 3, 2, 1, 1}$ arrangements of the 10 letters of four distinct types. All of them are equally likely, and only one spells statistics. So, the probability is $1/\binom{10}{3, 3, 2, 1, 1} = 1/50400$.

5. There are $\binom{n}{n_1, n_2, n_3, n_4, n_5, n_6}$ many ways to arrange n_j j's (for $j = 1, \ldots, 6$) among the n rolls. The number of possible equally likely rolls is 6^n. So, the probability we want is $\frac{1}{6^n}\binom{n}{n_1, n_2, n_3, n_4, n_5, n_6}$.

6. There are 6^7 possible outcomes for the seven dice. If each of the six numbers is to appear at least once among the seven dice, then one number must appear twice and each of the other five numbers must appear once. Suppose first that the number 1 appears twice and each of the other numbers appears once. The number of outcomes of this type in the sample space is equal to the number of different arrangements of the symbols 1, 1, 2, 3, 4, 5, 6, which is $\frac{7!}{2!(1!)^5} = \frac{7!}{2}$. There is an equal number of outcomes for each of the other five numbers which might appear twice among the seven dice. Therefore, the total number of outcomes in which each number appears at least once is $\frac{6(7!)}{2}$, and the probability of this event is

$$\frac{6(7!)}{(2)6^7} = \frac{7!}{2(6^6)}.$$

7. There are $\binom{25}{10,8,7}$ ways of distributing the 25 cards to the three players. There are $\binom{12}{6,2,4}$ ways of distributing the 12 red cards to the players so that each receives the designated number of red cards. There are then $\binom{13}{4,6,3}$ ways of distributing the other 13 cards to the players, so that each receives the designated total number of cards. The product of these last two numbers of ways is, therefore, the number of ways of distributing the 25 cards to the players so that each receives the designated number of red cards and the designated total number of cards. So, the final probability is $\binom{12}{6,2,4}\binom{13}{4,6,3}\Big/\binom{25}{10,8,7}$.

8. There are $\binom{52}{13,13,13,13}$ ways of distributing the cards to the four players. There are $\binom{12}{3,3,3,3}$ ways of distributing the 12 picture cards so that each player gets three. No matter which of these ways we choose, there are $\binom{40}{10,10,10,10}$ ways to distribute the remaining 40 nonpicture cards so that each player gets 10. So, the probability we need is

$$\frac{\binom{12}{3,3,3,3}\binom{40}{10,10,10,10}}{\binom{52}{13,13,13,13}} = \frac{\frac{12!}{(3!)^4}\frac{40!}{(10!)^4}}{\frac{52!}{(13!)^4}} \approx 0.0324.$$

9. There are $\binom{52}{13,13,13,13}$ ways of distributing the cards to the four players. Call these four players A, B, C, and D. There is only one way of distributing the cards so that player A receives all red cards, player B receives all yellow cards, player C receives all blue cards, and player D receives all green cards. However, there are 4! ways of assigning the four colors to the four players and therefore there are 4! ways of distributing the cards so that each player receives 13 cards of the same color. So, the probability we need is

$$\frac{4!}{\binom{52}{13,13,13,13}} = \frac{4!(13!)^4}{52!} \approx 4.474 \times 10^{-28}.$$

10. If we do not distinguish among boys with the same last name, then there are $\binom{9}{2,3,4}$ possible arrangements of the nine boys. We are interested in the probability of a particular one of these arrangements. So, the probability we need is

$$\frac{1}{\binom{9}{2,3,4}} = \frac{2!3!4!}{9!} \approx 7.937 \times 10^{-4}.$$

11. We shall use induction. Since we have already proven the binomial theorem, we know that the conclusion to the multinomial theorem is true for every n if $k = 2$. We shall use induction again, but this time using k instead of n. For $k = 2$, we already know the result is true. Suppose that the result is true for all $k \le k_0$ and for all n. For $k = k_0 + 1$ and arbitrary n we must show that

$$(x_1 + \cdots + x_{k_0+1})^n = \sum \binom{n}{n_1, \ldots, n_{k_0+1}} x_1^{n_1} \cdots x_{k_0+1}^{n_{k_0+1}}, \tag{S.1.4}$$

where the summation is over all $n_1, \ldots, n_{k_0+1}$ such that $n_1 + \cdots + n_{k_0+1} = n$. Let $y_i = x_i$ for $i = 1, \ldots, k_0 - 1$ and let $y_{k_0} = x_{k_0} + x_{k_0+1}$. We then have

$$(x_1 + \cdots + x_{k_0+1})^n = (y_1 + \cdots + y_{k_0})^n.$$

Since we have assumed that the theorem is true for $k = k_0$, we know that

$$(y_1 + \cdots + y_{k_0})^n = \sum \binom{n}{m_1, \ldots, m_{k_0}} y_1^{m_1} \cdots y_{k_0}^{m_{k_0}}, \tag{S.1.5}$$

where the summation is over all $m_1, \ldots, m_{k_0}$ such that $m_1 + \cdots + m_{k_0} = n$. On the right side of (S.1.5), substitute $x_{k_0} + x_{k_0+1}$ for y_{k_0} and apply the binomial theorem to obtain

$$\sum \binom{n}{m_1, \ldots, m_{k_0}} y_1^{m_1} \cdots y_{k_0-1}^{m_{k_0-1}} \sum_{i=0}^{m_{k_0}} \binom{m_{k_0}}{i} x_{k_0}^{i} x_{k_0+1}^{m_{k_0}-i}. \tag{S.1.6}$$

In (S.1.6), let $n_i = m_i$ for $i = 1, \ldots, k_0 - 1$, let $n_{k_0} = i$, and let $n_{k_0+1} = m_{k_0} - i$. Then, in the summation in (S.1.6), $n_1 + \cdots + n_{k_0+1} = n$ if and only if $m_1 + \cdots + m_{k_0} = n$. Also, note that

$$\binom{n}{m_1, \ldots, m_{k_0}} \binom{m_{k_0}}{i} = \binom{n}{n_1, \ldots, n_{k_0+1}}.$$

So, (S.1.6) becomes

$$\sum \binom{n}{n_1, \ldots, n_{k_0+1}} x_1^{n_1} \cdots x_{k_0+1}^{n_{k_0+1}},$$

where this last sum is over all $n_1, \ldots, n_{k_0+1}$ such that $n_1 + \cdots + n_{k_0+1} = n$.

12. For each element s' of S', the elements of S that lead to boxful s' are all the different sequences of elements of s'. That is, think of each s' as an unordered set of 12 numbers chosen with replacement from 1 to 7. For example, $\{1,1,2,3,3,3,5,6,7,7,7,7\}$ is one such set. The following are some of the elements of S lead to the same set s': $(1,1,2,3,3,3,5,6,7,7,7,7)$, $(1,2,3,5,6,7,1,3,7,3,7,7)$, $(7,1,7,2,3,5,7,1,6,3,7,3)$. This problem is pretty much the same as that which leads to the definition of multinomial coefficients. We are looking for the number of orderings of 12 digits chosen from the numbers 1 to 7 that have two of 1, one of 2, three of 3, none of 4, one of 5, one of 6, and four of 7. This is just $\binom{12}{1,1,3,0,1,1,4}$. For a general s', for $i = 1, \ldots, 7$, let $n_i(s')$ be the number of i's in the box s'. Then $n_1(s') + \cdots + n_7(s') = 12$, and the number of orderings of these numbers is

$$N(s') = \binom{12}{n_1(s'), n_2(s'), \ldots, n_7(s')}.$$

The multinomial theorem tells us that

$$\sum_{\text{All } s'} N(s') = \sum \binom{12}{n_1, n_2, \ldots, n_7} 1^{n_1} \cdots 1^{n_7} = 7^{12},$$

where the sum is over all possible combinations of nonnegative integers $n_1, \ldots, n_7$ that add to 12. This matches the number of outcomes in S.

1.10 The Probability of a Union of Events

Commentary

This section ends with an example of the matching problem. This is an application of the formula for the probability of a union of an arbitrary number of events. It requires a long line of argument and contains an interesting limiting result. The example will be interesting to students with good mathematics backgrounds, but it might be too challenging for students who have struggled to master combinatorics. One can use statistical software, such as *R*, to help illustrate how close the approximation is. The formula (1.10.10) can be computed as

```
ints=1:n
sum(exp(-lfactorial(ints))*(-1)^(ints+1)),
```

where `n` has previously been assigned the value of n for which one wishes to compute p_n.

Solutions to Exercises

1. Let A_i be the event that person i receives exactly two aces for $i = 1, 2, 3$. We want $\Pr(\cup_{i=1}^{3} A_i)$. We shall apply Theorem 1.10.1 directly. Let the sample space consist of all permutations of the 52 cards where the first five cards are dealt to person 1, the second five to person 2, and the third five to person 3. A permutation of 52 cards that leads to the occurrence of event A_i can be constructed as follows. First, choose which of person i's five locations will receive the two aces. There are $C_{5,2}$ ways to do this. Next, for each such choice, choose the two aces that will go in these locations, distinguishing the order in which they are placed. There are $P_{4,2}$ ways to do this. Next, for each of the preceding choices, choose the locations for the other two aces from among the 47 locations that are not dealt to person i, distinguishing order. There are $P_{47,2}$ ways to do this. Finally, for each of the preceding choices, choose a permutation of the remaining 48 cards among the remaining 48 locations. There are 48! ways to do this. Since there are 52! equally likely permutations in the sample space, we have

$$\Pr(A_i) = \frac{C_{5,2} P_{4,2} P_{47,2} 48!}{52!} = \frac{5!4!47!48!}{2!3!2!45!52!} \approx 0.0399.$$

Careful examination of the expression for $\Pr(A_i)$ reveals that it can also be expressed as

$$\Pr(A_i) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}.$$

This expression corresponds to a different, but equally correct, way of describing the sample space in terms of equally likely outcomes. In particular, the sample space would consist of the different possible five-card sets that person i could receive without regard to order.

Next, compute $\Pr(A_iA_j)$ for $i \neq j$. There are still $C_{5,2}$ ways to choose the locations for person i's aces amongst the five cards and for each such choice, there are $P_{4,2}$ ways to choose the two aces in order. For each of the preceding choices, there are $C_{5,2}$ ways to choose the locations for person j's aces and 2 ways to order the remaining two aces amongst the two locations. For each combination of the preceding choices, there are 48! ways to arrange the remaining 48 cards in the 48 unassigned locations. Then, $\Pr(A_iA_j)$ is

$$\Pr(A_iA_j) = \frac{2C_{5,2}^2 P_{4,2} 48!}{52!} = \frac{2(5!)^2 4! 48!}{(2!)^3(3!)^2 52!} \approx 3.694 \times 10^{-4}.$$

Once again, we can rewrite the expression for $\Pr(A_iA_j)$ as

$$\Pr(A_iA_j) = \frac{\binom{4}{2}\binom{48}{3,3,42}}{\binom{52}{5,5,42}}.$$

This corresponds to treating the sample space as the set of all pairs of five-card subsets.

Next, notice that it is impossible for all three players to receive two aces, so $\Pr(A_1A_2A_3) = 0$. Applying Theorem 1.10.1, we obtain

$$\Pr\left(\cup_{i=1}^3 A_i\right) = 3 \times 0.0399 - 3 \times 3.694 \times 10^{-4} = 0.1186.$$

2. Let A, B, and C stand for the events that a randomly selected family subscribes to the newspaper with the same name. Then $\Pr(A \cup B \cup C)$ is the proportion of families that subscribe to at least one newspaper. According to Theorem 1.10.1, we can express this probability as

$$\Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(AC) - \Pr(BC) + \Pr(A \cap BC).$$

The probabilities in this expression are the proportions of families that subscribe to the various combinations. These proportions are all stated in the exercise, so the formula yields

$$\Pr(A \cup B \cup C) = 0.6 + 0.4 + 0.3 - 0.2 - 0.1 - 0.2 + 0.05 = 0.85.$$

3. As seen from Fig. S.1.1, the required percentage is $P_1 + P_2 + P_3$. From the given values, we have, in percentages,

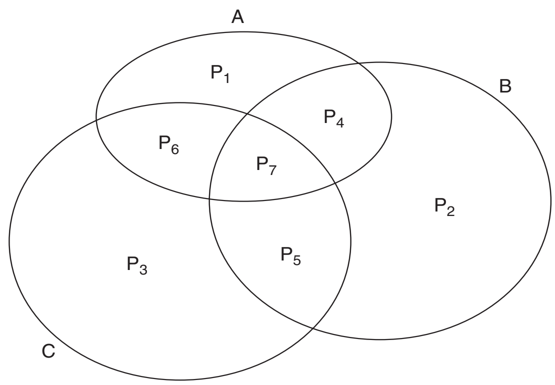

Figure S.1.1: Figure for Exercise 3 of Sec. 1.10.

$$\begin{aligned}
P_7 &= 5,\\
P_4 &= 20 - P_7 = 15,\\
P_5 &= 20 - P_7 = 15,\\
P_6 &= 10 - P_7 = 5,\\
P_1 &= 60 - P_4 - P_6 - P_7 = 35,\\
P_2 &= 40 - P_4 - P_5 - P_7 = 5,\\
P_3 &= 30 - P_5 - P_6 - P_7 = 5.
\end{aligned}$$

Therefore, $P_1 + P_2 + P_3 = 45$.

4. This is a case of the matching problem with $n = 3$. We are asked to find p_3. By Eq. (1.10.10) in the text, this equals

$$p_3 = 1 - \frac{1}{2} + \frac{1}{6} = \frac{2}{3}.$$

5. Determine first the probability that at least one guest will receive the proper hat. This probability is the value p_n specified in the matching problem, with $n = 4$, namely

$$p_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{5}{8}.$$

So, the probability that no guest receives the proper hat is $1 - 5/8 = 3/8$.

6. Let A_1 denote the event that no red balls are selected, let A_2 denote the event that no white balls are selected, and let A_3 denote the event that no blue balls are selected. The desired probability is $\Pr(A_1 \cup A_2 \cup A_3)$ and we shall apply Theorem 1.10.1. The event A_1 will occur if and only if the ten selected balls are either white or blue. Since there are 60 white and blue balls, out of a total of 90 balls, we have $\Pr(A_1) = \binom{60}{10}\Big/\binom{90}{10}$. Similarly, $\Pr(A_2)$ and $\Pr(A_3)$ have the same value. The event A_1A_2 will occur if and only if all ten selected balls are blue. Therefore, $\Pr(A_1A_2) = \binom{30}{10}\Big/\binom{90}{10}$. Similarly, $\Pr(A_2A_3)$ and $\Pr(A_1A_3)$ have the same value. Finally, the event $A_1A_2A_3$ will occur if and only if all three colors are missing, which is obviously impossible. Therefore, $\Pr(A_1A_2A_3) = 0$. When these values

are substituted into Eq. (1.10.1), we obtain the desired probability,

$$\Pr(A_1 \cup A_2 \cup A_3) = 3\frac{\binom{60}{10}}{\binom{90}{10}} - 3\frac{\binom{30}{10}}{\binom{90}{10}}.$$

7. Let A_1 denote the event that no student from the freshman class is selected, and let A_2, A_3, and A_4 denote the corresponding events for the sophomore, junior, and senior classes, respectively. The probability that at least one student will be selected from each of the four classes is equal to $1-\Pr(A_1 \cup A_2 \cup A_3 \cup A_4)$. We shall evaluate $\Pr(A_1 \cup A_2 \cup A_3 \cup A_4)$ by applying Theorem 1.10.2. The event A_1 will occur if and only if the 15 selected students are sophomores, juniors, or seniors. Since there are 90 such students out of a total of 100 students, we have $\Pr(A_1) = \binom{90}{15}\Big/\binom{100}{15}$. The values of $\Pr(A_i)$ for $i = 2, 3, 4$ can be obtained in a similar fashion. Next, the event A_1A_2 will occur if and only if the 15 selected students are juniors or seniors. Since there are a total of 70 juniors and seniors, we have $\Pr(A_1A_2) = \binom{70}{15}\Big/\binom{100}{15}$. The probability of each of the six events of the form A_iA_j for $i < j$ can be obtained in this way. Next the event $A_1A_2A_3$ will occur if and only if all 15 selected students are seniors. Therefore, $\Pr(A_1A_2A_3) = \binom{40}{15}\Big/\binom{100}{15}$. The probabilities of the events $A_1A_2A_4$ and $A_1A_3A_4$ can also be obtained in this way. It should be noted, however, that $\Pr(A_2A_3A_4) = 0$ since it is impossible that all 15 selected students will be freshmen. Finally, the event $A_1A_2A_3A_4$ is also obviously impossible, so $\Pr(A_1A_2A_3A_4) = 0$. So, the probability we want is

$$1 - \left[\frac{\binom{90}{15}}{\binom{100}{15}} + \frac{\binom{80}{15}}{\binom{100}{15}} + \frac{\binom{70}{15}}{\binom{100}{15}} + \frac{\binom{60}{15}}{\binom{100}{15}} \right.$$
$$\left. -\frac{\binom{70}{15}}{\binom{100}{15}} - \frac{\binom{60}{15}}{\binom{100}{15}} - \frac{\binom{50}{15}}{\binom{100}{15}} - \frac{\binom{50}{15}}{\binom{100}{15}} - \frac{\binom{40}{15}}{\binom{100}{15}} - \frac{\binom{30}{15}}{\binom{100}{15}} + \frac{\binom{40}{15}}{\binom{100}{15}} + \frac{\binom{30}{15}}{\binom{100}{15}} + \frac{\binom{20}{15}}{\binom{100}{15}}\right].$$

8. It is impossible to place exactly $n-1$ letters in the correct envelopes, because if $n-1$ letters are placed correctly, then the nth letter must also be placed correctly.

9. Let $p_n = 1 - q_n$. As discussed in the text, $p_{10} < p_{300} < 0.63212 < p_{53} < p_{21}$. Since p_n is smallest for $n = 10$, then q_n is largest for $n = 10$.

10. There is exactly one outcome in which only letter 1 is placed in the correct envelope, namely the outcome in which letter 1 is correctly placed, letter 2 is placed in envelope 3, and letter 3 is placed in envelope 2. Similarly there is exactly one outcome in which only letter 2 is placed correctly, and one in which only letter 3 is placed correctly. Hence, of the $3! = 6$ possible outcomes, 3 outcomes yield the result that exactly one letter is placed correctly. So, the probability is $3/6 = 1/2$.

11. Consider choosing 5 envelopes at random into which the 5 red letters will be placed. If there are exactly r red envelopes among the five selected envelopes ($r = 0, 1, \ldots, 5$), then exactly $x = 2r$ envelopes will

contain a card with a matching color. Hence, the only possible values of x are 0, 2, 4..., 10. Thus, for $x = 0, 2, \ldots, 10$ and $r = x/2$, the desired probability is the probability that there are exactly r red envelopes among the five selected envelopes, which is $\dfrac{\binom{5}{r}\binom{5}{5-r}}{\binom{10}{5}}$.

12. It was shown in the solution of Exercise 12 of Sec. 1.5. that

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \Pr(B_i) = \lim_{n\to\infty} \Pr\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \Pr\left(\bigcup_{i=1}^{n} A_i\right).$$

However, since $A_1 \subset A_2 \subset \ldots \subset A_n$, it follows that $\bigcup_{i=1}^{n} A_i = A_n$. Hence,

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n).$$

13. We know that

$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c.$$

Hence,

$$\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - \Pr\left(\bigcup_{i=1}^{\infty} A_i^c\right).$$

However, since $A_1 \supset A_2 \supset \ldots$, then $A_1^c \subset A_2^c \subset \ldots$. Therefore, by Exercise 12,

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i^c\right) = \lim_{n\to\infty} \Pr(A_n^c) = \lim_{n\to\infty} [1 - \Pr(A_n)] = 1 - \lim_{n\to\infty} Pr(A_n).$$

It now follows that

$$\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n).$$

1.12 Supplementary Exercises

Solutions to Exercises

1. No, since both A and B might occur.

2. $\Pr(A^c \cap B^c \cap D^c) = \Pr[(A \cup B \cup D)^c] = 0.3.$

3. $\dfrac{\binom{250}{18} \cdot \binom{100}{12}}{\binom{350}{30}}$.

4. There are $\binom{20}{10}$ ways of choosing 10 cards from the deck. For $j = 1, \ldots, 5$, there $\binom{4}{2}$ ways of choosing two cards with the number j. Hence, the answer is

$$\frac{\binom{4}{2} \cdots \binom{4}{2}}{\binom{20}{10}} = \frac{6^5}{\binom{20}{10}} \approx 0.0421.$$

5. The region where total utility demand is at least 215 is shaded in Fig. S.1.2. The area of the shaded

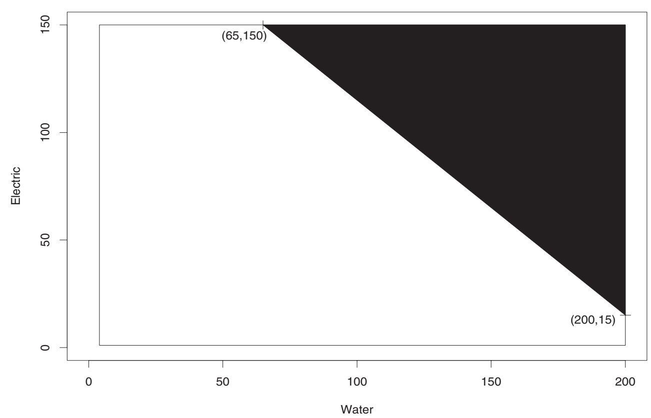

Figure S.1.2: Region where total utility demand is at least 215 in Exercise 5 of Sec. 1.12.

region is

$$\frac{1}{2} \times 135 \times 135 = 9112.5$$

The probability is then $9112.5/29204 = 0.3120$.

6. (a) There are $\binom{r+w}{r}$ possible positions that the red balls could occupy in the ordering as they are drawn. Therefore, the probability that they will be in the first r positions is $1/\binom{r+w}{r}$.

   (b) There are $\binom{r+1}{r}$ ways that the red balls can occupy the first $r+1$ positions in the ordering. Therefore, the probability is $\binom{r+1}{r}/\binom{r+w}{r} = (r+1)/\binom{r+w}{r}$.

7. The presence of the blue balls is irrelevant in this problem, since whenever a blue ball is drawn it is ignored. Hence, the answer is the same as in part (a) of Exercise 6.

8. There are $\binom{10}{7}$ ways of choosing the seven envelopes into which the red cards will be placed. There

are $\binom{7}{j}\binom{3}{7-j}$ ways of choosing exactly j red envelopes and $7-j$ green envelopes. Therefore, the probability that exactly j red envelopes will contain red cards is

$$\binom{7}{j}\binom{3}{7-j}\Big/\binom{10}{7} \qquad \text{for} \qquad j = 4, 5, 6, 7.$$

But if j red envelopes contain red cards, then $j-4$ green envelopes must also contain green cards. Hence, this is also the probability of exactly $k = j + (j-4) = 2j-4$ matches.

9. There are $\binom{10}{5}$ ways of choosing the five envelopes into which the red cards will be placed. There are $\binom{7}{j}\binom{3}{5-j}$ ways of choosing exactly j red envelopes and $5-j$ green envelopes. Therefore the probability that exactly j red envelopes will contain red cards is

$$\binom{7}{j}\binom{3}{5-j}\Big/\binom{10}{5} \qquad \text{for } j = 2, 3, 4, 5.$$

But if j red envelopes contain red cards, then $j-2$ green envelopes must also contain green cards. Hence, this is also the probability of exactly $k = j + (j-2) = 2j-2$ matches.

10. If there is a point x that belongs to neither A nor B, then x belongs to both A^c and B^c. Hence, A^c and B^c are not disjoint. Therefore, A^c and B^c will be disjoint if and only if $A \cup B = S$.

11. We can use Fig. S.1.1 by relabeling the events A, B, and C in the figure as A_1, A_2, and A_3 respectively. It is now easy to see that the probability that exactly one of the three events occurs is $p_1 + p_2 + p_3$. Also,

$$\begin{aligned}\Pr(A_1) &= p_1 + p_4 + p_6 + p_7,\\ \Pr(A_1 \cap A_2) &= p_4 + p_7, \text{ etc.}\end{aligned}$$

By breaking down each probability in the given expression in this way, we obtain the desired result.

12. The proof can be done in a manner similar to that of Theorem 1.10.2. Here is an alternative argument. Consider first a point that belongs to exactly one of the events $A_1, \ldots, A_n$. Then this point will be counted in exactly one of the $\Pr(A_i)$ terms in the given expression, and in none of the intersections. Hence, it will be counted exactly once in the given expression, as required. Now consider a point that belongs to exactly r of the events $A_1, \ldots, A_n (r \geq 2)$. Then it will be counted in exactly r of the $\Pr(A_i)$ terms, exactly $\binom{r}{2}$ of the $\Pr(A_iA_j)$ terms, exactly $\binom{r}{3}$ of the $\Pr(A_iA_jA_k)$ terms, etc. Hence, in the given expression it will be counted the following number of times:

$$\begin{aligned} r &- 2\binom{r}{2} + 3\binom{r}{3} - \cdots \pm r\binom{r}{r}\\ &= r\left[\binom{r-1}{0} - \binom{r-1}{1} + \binom{r-1}{2} - \cdots \pm \binom{r-1}{r-1}\right] = 0,\end{aligned}$$

by Exercise b of Sec. 1.8. Hence, a point will be counted in the given expression if and only if it belongs to exactly one of the events $A_1, \ldots, A_n$, and then it will be counted exactly once.

13. (a) In order for the winning combination to have no consecutive numbers, between every pair of numbers in the winning combination there must be at least one number not in the winning combination. That is, there must be at least $k-1$ numbers not in the winning combination to be in between the pairs of numbers in the winning combination. Since there are k numbers in the winning combination, there must be at least $k+k-1=2k-1$ numbers available in order for it to be possible to have no consecutive numbers in the winning combination. So, n must be at least $2k-1$ to allow consecutive numbers.

(b) Let $i_1,\ldots,i_k$ and $j_1,\ldots,j_k$ be as described in the problem. For one direction, suppose that $i_1,\ldots,i_k$ contains at least one pair of consecutive integers, say $i_{a+1}=i_a+1$. Then

$$j_{a+1}=i_{a+1}-a=i_a+1-a=i_a-(a-1)=j_a.$$

So, $j_1,\ldots,j_k$ contains repeats. For the other direction, suppose that $j_1,\ldots,j_k$ contains repeats, say $j_{a+1}=j_a$. Then

$$i_{a+1}=j_{a+1}+a=j_a+a=i_a+1.$$

So $i_1,\ldots,i_k$ contains a pair of consecutive numbers.

(c) Since $i_1<i_2<\cdots<i_k$, we know that $i_a+1\le i_{a+1}$, so that $j_a=i_a-a+1\le i_{a+1}-a=j_{a+1}$ for each $a=1,\ldots,k-1$. Since $i_k\le n$, $j_k=i_k-k+1\le n-k+1$. The set of all $(j_1,\ldots,j_k)$ with $1\le j_1<\cdots<j_k\le n-k+1$ is just the number of combinations of $n-k+1$ items taken k at a time, that is $\binom{n-k+1}{k}$.

(d) By part (b), there are no pairs of consecutive integers in the winning combination $(i_1,\ldots,i_k)$ if and only if $(j_1,\ldots,j_k)$ has no repeats. The total number of winning combinations is $\binom{n}{k}$. In part (c), we computed the number of winning combinations with no repeats among $(j_1,\ldots,j_k)$ to be $\binom{n-k+1}{k}$. So, the probability of no consecutive integers is

$$\frac{\binom{n-k+1}{k}}{\binom{n}{k}}=\frac{(n-k)!(n-k+1)!}{n!(n-2k+1)!}.$$

(e) The probability of at least one pair of consecutive integers is one minus the answer to part (d).

Chapter 2

Conditional Probability

2.1 The Definition of Conditional Probability

Commentary

It is useful to stress the point raised in the note on page 59. That is, conditional probabilities behave just like probabilities. This will come up again in Sec. 3.6 where conditional distributions are introduced.

This section ends with an extended example called "The Game of Craps". This example helps to reinforce a subtle line of reasoning about conditional probability that was introduced in Example 2.1.5. In particular, it uses the idea that conditional probabilities given an event B can be calculated as if we knew ahead of time that B had to occur.

Solutions to Exercises

1. If $A \subset B$, then $A \cap B = A$ and $\Pr(A \cap B) = \Pr(A)$. So $\Pr(A|B) = \Pr(A)/\Pr(B)$.

2. Since $A \cap B = \emptyset$, it follows that $\Pr(A \cap B) = 0$. Therefore, $\Pr(A \mid B) = 0$.

3. Since $A \cap S = A$ and $\Pr(S) = 1$, it follows that $\Pr(A \mid S) = \Pr(A)$.

4. Let A_i stand for the event that the shopper purchases brand A on his ith purchase, for $i = 1, 2, \ldots$. Similarly, let B_i be the event that he purchases brand B on the ith purchase. Then

$$\begin{aligned}
\Pr(A_1) &= \frac{1}{2}, \\
\Pr(A_2 \mid A_1) &= \frac{1}{3}, \\
\Pr(B_3 \mid A_1 \cap A_2) &= \frac{2}{3}, \\
\Pr(B_4 \mid A_1 \cap A_2 \cap B_3) &= \frac{1}{3}.
\end{aligned}$$

   The desired probability is the product of these four probabilities, namely $1/27$.

5. Let R_i be the event that a red ball is drawn on the ith draw, and let B_i be the event that a blue ball is drawn on the ith draw for $i = 1, \ldots, 4$. Then

$$\Pr(R_1) = \frac{r}{r+b},$$